Lecture notes

C*-algebras and K-theory

Matthias Ludewig
SS 2020

version of April 18, 2024

If you find any mistakes (even spelling mistakes), please tell me!

Contents

1 Some theory of C^*-algebras

In this section, we give the definition of Banach algebras, C^*-algebras and related notions. Then we review the spectral theory of Banach and C^*-algebras. Finally, we prove the Gelfand-Naimark theorem and the existence of a continuous functional calculus for normal elements.

1.1 Basic definitions

Definition 1.1 (Banach and C^* algebras)**.** All algebras are over the field $\mathbb{C}$.

(a) A *Banach algebra* is an algebra A together with a norm that turns it into a Banach space and is *submultiplicative*, that is

$$\|ab\| \leq \|a\|\|b\| \qquad \forall a, b \in A.$$

(b) A $*$-algebra is an algebra A together with an anti-linear involution $A \to A$, $a \mapsto a^*$, the $*$-*operation*, such that

$$(ab)^* = b^* a^* \qquad \forall a, b \in A.$$

(c) A C^*-algebra is a Banach algebra with a $*$-operation satisfying the C^*-*property*

$$\|a^* a\| = \|a\|^2 \qquad \forall a \in A. \tag{1}$$

(d) A Banach algebra A is *unital* if A has a unit $\mathbf{1} \neq 0$ such that $\|\mathbf{1}\| = 1$. (For C^*-algebras, this follows from (1).)

The following facts will be used throughout without mentioning.

Lemma 1.2. In any (unital) C^*-algebra, we have

$$\|a\| = \|a^*\| \qquad \text{and} \qquad \mathbf{1}^* = \mathbf{1}. \tag{2}$$

Proof. To see the first identity, calculate $\|a\|^2 = \|a^* a\| \leq \|a^*\|\|a\|$, hence $\|a\| \leq \|a^*\|$; replacing a by a^* yields $\|a^*\| \leq \|a\|$. To see the second identity, observe that $\mathbf{1}^* a = (a^* \mathbf{1})^* = (a^*)^* = \mathbf{1}$ for all $a \in A$; therefore $\mathbf{1} = \mathbf{1}^* \mathbf{1} = \mathbf{1}^*$. □

Example 1.3 (Continuous functions)**.** Let X be a locally compact Hausdorff space. The space

$$C_0(X) = \{f : X \to \mathbb{C} \text{ continuous} \mid \forall \varepsilon > 0 \exists K \subset X \text{ compact} : \|f|_{X\setminus K}\|_\infty < \varepsilon\}$$

is a commutative Banach algebra with pointwise multiplication and the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. It is even a C^*-algebra with $f^*(x) = \overline{f(x)}$ (pointwise complex conjugation). $C_0(X)$ is unital if and only if X is compact. In that case, $C_0(X) = C(X)$, the algebra of continuous functions on X.

Example 1.4 (Algebras of operators)**.** Let H be a complex Hilbert space.

(1) $\mathbb{B}(H)$, the algebra of bounded operators on H, is a unital Banach algebra with respect to the operator norm. It is commutative precisely when $\dim(H) \leq 1$. It is even a $*$-algebra with respect to taking the adjoint map.

(2) $\mathbb{K}(H)$, the algebra of compact operators on H, is a closed subalgebra of $\mathbb{B}(H)$. Since taking adjoints preserves $\mathbb{K}(H)$, it is a C^*-algebra. It is unital precisely when H is finite-dimensional and commutative precisely when $\dim(H) \leq 1$.

Example 1.5 (Direct sums)**.** If A, B are Banach algebras (C^*-algebras), their direct sum $A \oplus B$ is a Banach algebra (C^*-algebra) with the norm $\|(a, b)\| := \sup\{\|a\|, \|b\|\}$.

Example 1.6 (Quotients)**.** If A is a Banach algebra an $J \subset A$ is a *closed* ideal, then A/J is a Banach algebra with the norm $\|[a]\| = \inf\{\|a + b\| \mid b \in J\}$. If A is a C^*-algebra, things are slightly involved. First of all, it is a fact that any closed ideal J of A is automatically $*$-closed, i.e. $a^* \in J$ for all $a \in J$ [3, Thm. 3.1.3]. Therefore, the $*$-operation $[a]^* := [a^*]$ is well-defined on A/J. Hence A/J is a $*$-algebra, but showing the C^*-identity $\|[a]\|^2 = \|[a]^*[a]\|$ is rather tricky; one way to see this is via approximate units (see e.g. [3, Thm. 3.1.4] or [2, Prop. 1.8.2]). For another approach, see [4, Exercise 1.B].

Example 1.7 (Matrix algebras). If A is an algebra and $n \in \mathbb{N}$, the algebra $M_n(A)$ the algebra of $n \times n$ matrices with entries in A is an algebra with matrix multiplication. If A is $*$-algebra, there is a $*$-operation on $M_n(A)$ given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & \cdots & a_{n1}^* \\ \vdots & \ddots & \vdots \\ a_{1n}^* & \cdots & a_{nm}^* \end{pmatrix},$$

turning also $M_n(A)$ into a $*$-algebra. If A is moreover a C^*-algebra, Lemma 1.38 shows that there is a suitable norm on $M_n(A)$ turning it into a C^*-algebra.

Definition 1.8 (Homomorphisms). Let A, B be algebras.

(a) A *homomorphism* $\Phi : A \to B$ is a linear map which is multiplicative, that is $\Phi(ab) = \Phi(a)\Phi(b)$ for $a, b \in A$.

(b) If A and B are unital, then we say that Φ is *unital* if $\Phi(\mathbf{1}_A) = \mathbf{1}_B$.

(c) If A and B are $*$-algebras, a *$*$-homomorphism* is a homomorphism $\Phi : A \to B$ such that $\Phi(a^*) = \Phi(a)^*$ for all $a \in A$.

Remark 1.9. If A, B are Banach algebras, we do *not* require continuity of homomorphisms $\Phi : A \to B$. Instead, continuity is often automatic (see e.g. Prop. 1.16 and Lemma 1.22).

Example 1.10. If B is another C^*-algebra and $\Phi : A \to B$ a $*$-homomorphism, we get an induced $*$-homomorphism $M_n(\Phi) : M_n(A) \to M_n(B)$, which is given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} \Phi(a_{11}) & \cdots & \Phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \Phi(a_{n1}) & \cdots & \Phi(a_{nn}) \end{pmatrix}. \tag{3}$$

Clearly, associating matrix algebras assembles to a functor that sends the category of C^*-algebras and $*$-homomorphisms to itself. For convenience, we will usually just write again Φ instead of $M_n(\Phi)$.

1.2 Spectral theory

Definition 1.11. Let A be a unital Banach algebra and let $a \in A$.

(a) $\rho(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \text{ invertible}\}$ is called *resolvent set* of a.

(b) $\sigma(a) = \mathbb{C} \setminus \rho(a)$ is called *spectrum* of a.

If A is moreover a C^*-algebra, then

(c) a is called *normal* if $a^*a = aa^*$.

(d) a is called *self-adjoint* if $a = a^*$.

Example 1.12. If X is a compact Hausdorff space, then for $f \in C(X)$, we have $\sigma(f) = \{f(x) \mid x \in X\}$. This follows directly from the fact that $f \in C(X)$ is invertible precisely if $f(x) \neq 0$ for all $x \in X$.

Example 1.13. Let H be a Hilbert space and $T \in \mathbb{B}(H)$. The *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ of T consists of those $\lambda \in \mathbb{C}$ such that $\lambda - T$ is not a Fredholm operator. Remember here that T is called *Fredholm* if it has closed range and finite-dimensional kernel and cokernel; equivalently (by *Atkinsons's theorem*), it is one that admits a *parametrix*, that is an operator S such that $TS - \mathrm{id}_H, ST - \mathrm{id}_H \in \mathbb{K}(H)$. Clearly, the latter condition is equivalent to saying that the class $[T]$ is invertible in $\mathbb{B}(H)/\mathbb{K}(H)$. We conclude that $\sigma_{\mathrm{ess}}(T) = \sigma([T])$, the spectrum of $[T]$ in $\mathbb{B}(H)/\mathbb{K}(H)$

Proposition 1.14. Let A be a unital Banach algebra and let $a \in A$.

(a) $\rho(a)$ is open.

(b) $\sigma(a)$ is compact, more precisely, $|\lambda| \leq \|a\|$ for all $\lambda \in \sigma(a)$.

(c) $\sigma(a) \neq \emptyset$.

If A is moreover a C^*-algebra, then

(d) if a is normal, then $\|a\|^2 = \|a^2\|$ and $\|a\| = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$;

(e) if a is self-adjoint, then $\sigma(a) \subseteq \mathbb{R}$.

(f) $\sigma(a^*a) \subset \mathbb{R}_{\geq 0}$;

Proof. Except for (f), these results are proven just as the special case $A = \mathbb{B}(H)$. Assertion (f) is non-trivial; a proof can be found in [3, §2.2]. □

Proposition 1.15. Any $*$-algebra has at most one submultiplicative norm satisfying the C^*-property with respect to which it is complete.

Proof. This follows from the fact that the norm is determined by the algebra structure: For all $a \in A$,

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda| \mid \lambda \in \sigma(a^*a)\},$$

where the first equality is the C^*-property and the second equality follows from Prop. 1.14(d), as a^*a is normal. □

Proposition 1.16. Let A and B be a unital C^*-algebras. Then any unital $*$-homomorphism $\Phi : A \to B$ is contractive, i.e. $\|\Phi(a)\| \leq \|a\|$ for all $a \in A$.

Proof. Let $a \in A$ and $\lambda \in \rho(a)$. Then $\lambda - a$ is invertible, say $(\lambda - a)b = \mathbf{1}_A$. Since Φ is unital,

$$\mathbf{1}_B = \Phi(\mathbf{1}_A) = \Phi((\lambda - a)b) = (\lambda - \Phi(a))\Phi(b).$$

we conclude that $\lambda - \Phi(a)$ is invertible, with inverse $\Phi(b)$, hence $\lambda \in \rho(\Phi(a))$. Hence $\rho(a) \subseteq \rho(\Phi(a))$ and $\sigma(\Phi(a)) \subseteq \sigma(a)$. Now by the C^*-property and Prop. 1.14(d),

$$\begin{aligned}\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| &= \sup\{|\lambda| \mid \lambda \in \sigma(\Phi(a)^*\Phi(a)))\} \\ &\leq \sup\{|\lambda| \mid \lambda \in \sigma(a^*a))\} = \|a^*a\| = \|a\|^2.\end{aligned}$$

This finishes the proof. □

Theorem 1.17 (Gelfand-Mazur)**.** Let A be a unital Banach algebra where every element $0 \neq a \in A$ is invertible. Then A is one-dimensional.

Proof. Let $a \in A$. Since $\sigma(a) \neq \emptyset$ (Prop. 1.14(c)), we can choose $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. Hence by assumption on A, $a - \lambda = 0$, i.e. a is a multiple of the identity. □

1.3 The unitalization

Definition 1.18. Let A be a C^*-algebra. Its *unitalization* A^+ is the unital $*$-algebra with underlying vector space $A^+ = A \oplus \mathbb{C}$, product

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu).$$

and $*$-operation

$$(a, \lambda)^* := (a^*, \overline{\lambda}).$$

That the product and $*$-operation given above indeed turn A^+ into a $*$-algebra is a straightforward calculation. The identification $a \mapsto (a, 0)$ embeds A into A^+, and we just write $A \subset A^+$. One easily checks that A is an ideal in A^+. In fact, it is the kernel of the *augmentation map*, which is the $*$-homomorphism

$$\varepsilon_A : A^+ \longrightarrow \mathbb{C}, \qquad \varepsilon_A(a, \lambda) = \lambda.$$

Proposition 1.19. For any C^*-algebra A, the $*$-algebra A^+ is a C^*-algebra, that is, there exists a unique norm of A^+ that turns A^+ into a C^*-algebra.

Proof. We have to prove the existence; uniqueness then follows from Prop. 1.15. If A is unital, we have $A^+ \cong A \oplus \mathbb{C}$ as $*$-algebras (via the isomorphism $(a, \lambda) \mapsto (a + \lambda \mathbf{1}_A, \lambda))$ and $A \oplus \mathbb{C}$ is a C^*-algebra.
Assume now that A is non-unital. In this case, we define a norm by

$$\|(a, \lambda)\| := \sup_{\|b\| \leq 1} \|ab + \lambda b\|, \tag{4}$$

where the supremum is taken over all $b \in A$ with $\|b\| \leq 1$ and the norm on the right hand side is the norm of A. The norm is definite because $\|ab + \lambda b\| = 0$ for all $b \in A$ would mean that either $\lambda = 0$ and hence $a = 0$ or $-\lambda^{-1}a$ is a unit for A. It is submultiplicative because

$$\begin{aligned} \|(a, \lambda) \cdot (b, \mu)\| &= \sup_{\|c\| \leq 1} \|abc + \lambda bc + \mu ac + \lambda\mu c\| \\ &= \sup_{bc + \mu c \neq 0} \frac{\|a(bc + \mu c) + \lambda(bc + \mu c)\|}{\|bc + \mu c\|} \|bc + \mu c\| \\ &\leq \sup \|d\| \leq 1 \|ad + \lambda d\| \cdot \sup_{\|c\| \leq 1} \|ac + \lambda c\| = \|(a, \lambda)\| \|(b, \mu)\|. \end{aligned}$$

We verify that the norm verifies the C^*-identity. To this end, we first observe that in any C^*-algebra, we have the identity

$$\|a\| = \sup_{\|b\|\leq 1} \|ab\| = \sup_{\|b\|\leq 1} \|ba\|, \tag{5}$$

since on the one hand, if $\|b\| \leq 1$, then $\|ab\| \leq \|a\|\|b\| \leq \|a\|$ by submultiplicativity, and on the other hand, the C^*-identity implies $\|a\| = \|ab\|$ for $b = \|a\|^{-1}a^*$. Now

$$\begin{aligned}
\|(a,\lambda)^*(a,\lambda)\| &= \|(a^*a + \overline{\lambda}a + \lambda a^*, |\lambda|^2)\| \\
&= \sup_{\|b\|\leq 1} \|(a^*a + \overline{\lambda}a + \lambda a^*)b + |\lambda|^2 b\| \\
&= \sup_{\|c\|\leq 1} \sup_{\|b\|\leq 1} \|c(a^*a + \overline{\lambda}a + \lambda a^*)b + c|\lambda|^2 b\| \\
&\geq \sup_{\|b\|\leq 1} \|b^*(a^*a + \overline{\lambda}a + \lambda a^*)b + b^*|\lambda|^2 b\| \\
&= \sup_{\|b\|\leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\
&= \sup_{\|b\|\leq 1} \|ab + \lambda b\|^2 \\
&= \|(a,\lambda)\|^2,
\end{aligned}$$

where we used the C^*-identity of the norm of A. The inequality $\|(a,\lambda)^*(a,\lambda)\| \leq \|(a,\lambda)\|^2$ follows from submultiplicativity.
Completeness follows from the following lemma from functional analysis: *If V is a normed vector space and $W \subset V$ is a subspace of codimension one which is complete with respect to the induced norm, then V is also complete.* Indeed, $W := A$ has codimension one in $V := A^+$, and the norm that (4) induces on W its original norm by (5). Hence by the lemma, $V = A^+$ is complete. □

Remark 1.20. The obvious norm $\|(a,\lambda)\|_1 := \|a\| + |\lambda|$ turns A^+ into a Banach algebra, but it does not satisfy the C^*-property. However, by (4), we have $\|(a,\lambda)\| \leq \|(a,\lambda)\|_1$. Hence the identity map from A^+ with the norm $\|\cdot\|_1$ to A^+ with the norm (4) is bounded. It then follows from the open mapping theorem that also its inverse is bounded, hence both norms are equivalent.

If A, B are C^*-algebras and $\Phi : A \to B$ is a $*$-homomorphism, we obtain a unital $*$-homomorphism

$$\Phi^+ : A^+ \longrightarrow B^+, \qquad \Phi^+(a,\lambda) = (\Phi(a),\lambda).$$

It is straightforward to check that $A \mapsto A^+$, $\Phi \mapsto \Phi^+$ defines a functor (the *unitalization functor*) from the category of C^*-algebras and $*$-homomorphisms to the category of unital C^*-algebras and unital $*$-homomorphisms. In fact, since $\varepsilon_B \circ \Phi = \varepsilon_A$, the target category of the unitalization functor is the category of *augmented* unital C^*-algebras together with compatible unital $*$-homomorphisms. This observation motivates the definition of the K_0-functor below.

Remark 1.21. The unitalization of a C^*-algebra A has the following universal property. For any unital C^*-algebra B and any $*$-homomorphism $\Phi : A \to B$ (not necessarily unital if A is unital), there exists a unique *unital* $*$-homomorphism $\Phi^+ : A^+ \to B$ that restricts to Φ on $A \subset A^+$. Hence the unitalization is a left adjoint to the forgetful functor from the category of unital C^*-algebra and unital $*$-homomorphisms to the category of C^*-algebras and $*$-homomorphisms.

1.4 The Gelfand-Naimark theorem

Lemma 1.22. Let A be a Banach algebra. Then every homomorphism $\varphi : A \to \mathbb{C}$ is continuous with $\|\varphi\| \leq 1$. If A is unital, we have more precisely either $\|\varphi\| = 1$ or $\varphi = 0$.

Proof. Suppose that $1 < \|\varphi\| \leq \infty$. Then there exists $a \in A$ with $|\varphi(a)| > \|a\|$. Set $a' := \varphi(a)^{-1}a$; then $\varphi(a') = 1$, but $\|a'\| = |\varphi(a)|^{-1}\|a\| < 1$. Set $b = \sum_{n=1}^{\infty}(a')^n$, where the series converges absolutely as $\|a'\| < 1$. Then $b = a' + a'b$ and therefore

$$\varphi(b) = \varphi(a') + \varphi(a')\varphi(b) = 1 + \varphi(b),$$

a contradiction. Hence $\|\varphi\| \leq 1$.
If now $\varphi \neq 0$, then there exists $a \in A$ with $\varphi(a) \neq 0$. Again setting $a' = \varphi(a)^{-1}a$, we have $\varphi(a') = 1$. Therefore, if A has a unit,

$$\|\mathbf{1}\| = 1 = \varphi(a') = \varphi(a'\mathbf{1}) = \underbrace{\varphi(a')}_{=1}\varphi(\mathbf{1}) = \varphi(\mathbf{1}),$$

hence $\|\varphi\| \geq 1$. $\square$

Definition 1.23. For a Banach algebra A, the *Gelfand space* is the set

$$\Gamma_A := \{\varphi : A \to \mathbb{C} \text{ homomorphism}, \varphi \neq 0\}.$$

By Lemma 1.22, each $\varphi \in \Gamma_A$ is in fact continuous, i.e. an element of the dual space A' of the Banach space A. Remember that A' carries the weak-$*$-topology, which can be characterized as the coarsest topology such that for each $a \in A$, the linear functional $\widehat{a} : A' \to \mathbb{C}$, $\widehat{a}(\varphi) = \varphi(a)$ is continuous.

Lemma 1.24. Let A be a unital Banach algebra and $J \subset A$ a maximal ideal. Then J is closed.

Proof. Let J be a maximal ideal (in particular proper!). Then, by continuity of the multiplication, its closure $\overline{J}$ is again an ideal. Since $J \subset \overline{J}$ and J is maximal, we have either $\overline{J} = J$ or $\overline{J} = A$. The latter means that J is dense in A; we show that this is not possible. Namely, any $a \in A$ with $\|\mathbf{1} - a\| < 1$ is invertible, with inverse given by the Neumann series $a^{-1} = \sum_{n=0}^{\infty}(\mathbf{1} - a)^n$. On the other hand, if J is dense, it must have a non-trivial intersection with the open set $\{a \mid \|\mathbf{1} - a\| < 1\}$, hence contain an invertible element. But this would imply $J = A$, which is impossible since J is a proper ideal. □

Proposition 1.25. Let A be a unital commutative Banach algebra.

(a) Γ_A equipped with the weak-$*$-topology is a compact Hausdorff space, and for each $a \in A$, we have $\widehat{a} \in C(\Gamma_A)$.

(b) Every maximal ideal $J \subset A$ is of the form $J = \ker(\varphi)$ for $\varphi \in \Gamma_A$.

(c) We have $\Gamma_A \neq \emptyset$. More precisely, $\sigma(a) = \{\varphi(a) \mid \varphi \in \Gamma_A\}$ for all $a \in A$.

(d) For all $a \in A$, $\sigma(a) = \sigma(\widehat{a})$.

Proof. (a) We have

$$\Gamma_A = \bigcap_{a,b \in A} \{\varphi \in A' \mid \varphi(ab) - \varphi(a) - \varphi(b) = 0\} \cap \{\varphi \in A' \mid \varphi(\mathbf{1}) = 1\}.$$

Since the maps $A' \to \mathbb{C}$, $\varphi \mapsto \varphi(a)$ are weak-$*$-continuous for each $a \in A$ (by the above characterization of the topology), we see that Γ_A is closed. On the other hand Lemma 1.22, we have $\|\varphi\| = 1$ for each $\varphi \in \Gamma_A$, hence Γ_A is a subset of the unit ball of A', which is compact with respect to the weak-$*$-topology, by the Banach-Alaoglu theorem. We conclude that Γ_A is a closed subset of a compact set, hence compact. The $\widehat{a}$ are continuous, again by the characterization of the weak-$*$-topology.
(b) It is a general fact that for a commutative ring A, the quotient A/J by an ideal J is a field if and only if J is maximal. Suppose that J is a maximal ideal, so that A/J is a field. Now, any maximal ideal in a Banach algebra is closed by Lemma 1.24, and the quotient of a Banach algebra by a closed ideal is again a Banach algebra (see Example 1.6). From Thm. 1.17, we therefore get $A/J \cong \mathbb{C}$. The quotient map $\varphi : A \to A/J \cong \mathbb{C}$ is a homomorphism with $\ker(\varphi) = J$. Conversely, since for $\varphi \in \Gamma_A$, the quotient $A/\ker(\varphi) \cong \mathbb{C}$ is a field, $\ker(\varphi)$ must be a maximal ideal.
(c) First $\Gamma_A \neq \emptyset$ by (b), as any ideal is contained in a maximal ideal (Zorn's lemma!). Suppose that $\lambda \notin \sigma(a)$, so that $\lambda - a$ is invertible. Then for every $\varphi \in \Gamma_A$, $\varphi(\lambda - a) = \lambda - \varphi(a) \in \mathbb{C}$ is non-zero by multiplicativity of φ. Hence $\lambda \notin \{\varphi(a) \mid \varphi \in \Gamma_A\}$. This shows that $\{\varphi(a) \mid a \in A\} \subseteq \sigma(a)$. Conversely, suppose that $\lambda \in \sigma(a)$. Then $J = \{(\lambda - a)b \mid b \in A\}$ is an ideal. It is proper since $\mathbf{1} \in J$ would imply $\mathbf{1} = (\lambda - a)b$ for some $b \in A$, hence $b = (\lambda - a)^{-1}$, a contradiction. Therefore, J is contained

in a maximal ideal, which by (b) is of the form $\ker(\varphi)$ with $\varphi \in \Gamma_A$. We obtain $\varphi((\lambda - a)b) = 0$ for each $b \in A$; in particular for $b = \mathbf{1}$, we get $\varphi(a) = \lambda$, hence $\sigma(a) \subseteq \{\varphi(a) \mid \varphi \in \Gamma_A\}$.
(d) Since $\widehat{a} \in C(\Gamma_A)$, Example 1.12 gives

$$\sigma(\widehat{a}) = \{\widehat{a}(\varphi) \mid \varphi \in \Gamma_A\} = \{\varphi(a) \mid \varphi \in \Gamma_A\}.$$

But this equals $\sigma(a)$ by (c). □

Theorem 1.26 (Gelfand representation)**.** Let A be a commutative unital Banach algebra. Then $A \to C(\Gamma_A)$, $a \mapsto \widehat{a}$, called *Gelfand transform*, is a unital homomorphism with

$$\|\widehat{a}\|_\infty = \sup\{|\lambda| \mid \lambda \in \sigma(a)\} \leq \|a\|.$$

Proof. We have

$$(\widehat{a}\widehat{b})(\varphi) = \widehat{a}(\varphi)\widehat{b}(\varphi) = \varphi(a)\varphi(b) = \varphi(ab) = \widehat{ab}(\varphi), \qquad \widehat{e}(\varphi) = \varphi(e) = 1.$$

Hence the Gelfand transform is a unital algebra homomorphism. Moreover,

$$\|\widehat{a}\|_\infty = \sup\{|\widehat{a}(\varphi)| \mid \varphi \in \Gamma_A\} = \sup\{|\varphi(a)| \mid \varphi \in \Gamma_A\} = \sup\{|\lambda| \mid \lambda \in \sigma(a)\},$$

where in the last step, we used Prop. 1.25(c). By Prop. 1.14(b), this is estimated by $\|a\|$. □

Theorem 1.27 (Gelfand-Naimark)**.** Let A be a commutative unital C^*-algebra. Then the Gelfand transform $A \to C(\Gamma_A)$, $a \mapsto \widehat{a}$ is an isometric $*$-isomorphism.

Proof. We show that the Gelfand transform is a $*$-homomorphism, i.e. $\widehat{a^*} = \overline{\widehat{a}}$. First let a be self-adjoint, for which we have to show that $\widehat{a}$ is real. Because a is self-adjoint, we have $\sigma(a) \subset \mathbb{R}$ (Prop. 1.14(e)). But by Prop. 1.25(d) and Example 1.12, we have

$$\mathbb{R} \supset \sigma(a) = \sigma(\widehat{a}) = \{\widehat{a}(\varphi) \mid \varphi \in \Gamma_A\}.$$

Hence $\widehat{a}$ is real-valued. The general case follows from writing

$$a = b + ic = \frac{1}{2}(a + a^*) + i\frac{1}{2i}(a - a^*).$$

Then b and c are self-adjoint and by the previous step,

$$\widehat{a^*} = \widehat{b^*} - i\widehat{c^*} = \overline{\widehat{b}} - i\overline{\widehat{c}} = \overline{\widehat{b} + i\widehat{c}} = \overline{\widehat{a}}.$$

We show that the Gelfand transform is isometric. Since A is commutative, any $a \in A$ is normal, hence

$$\begin{aligned} \|a\| &= \sup\{|\lambda| \mid \lambda \in \sigma(a)\} && \text{(Prop. 1.14(d))} \\ &= \sup\{|\lambda| \mid \lambda \in \sigma(\widehat{a})\} && \text{(Prop. 1.25(d))} \\ &= \sup\{\widehat{a}(\varphi) \mid \varphi \in \Gamma_A\} && \text{(Example 1.12)} \\ &= \|\widehat{a}\|_\infty. \end{aligned}$$

We use the theorem of Stone-Weierstraß to show that $\widehat{A} \subseteq C(\Gamma_A)$ is dense. To this end, we have to show that

(i) $\widehat{A}$ seperates points, i.e. if for $\varphi, \psi \in \Gamma_A$, we have $\widehat{a}(\varphi) = \widehat{a}(\psi)$ for all $a \in A$, then $\varphi = \psi$. But this is clear since $\varphi = \psi \in A'$ if $\varphi(a) = \psi(a)$ for all $a \in A$.

(ii) No evaluation functional vanishes, i.e. for all $\varphi \in \Gamma_A$, there exists $\widehat{a} \in \widehat{A}$ with $\widehat{a}(\varphi) \neq 0$. Again, this is clear, because if for some $\varphi \in \Gamma$, one has $\widehat{a}(\varphi) = \varphi(a) = 0$ for all $a \in A$, then $\varphi = 0$, hence $\varphi \notin \Gamma_A$.

(iii) $\widehat{A}$ is closed under complex conjugation. But this follows since for $\widehat{a} \in \widehat{A}$, $\overline{\widehat{a}} = \widehat{a^*} \in \widehat{A}$.

We conclude that $\widehat{A}$ is dense in $C(\Gamma_A)$. But since $a \mapsto \widehat{a}$ is isometric, $\widehat{A}$ is also closed, hence $\widehat{A} = C(\Gamma_A)$. □

Remark 1.28. It follows from the proof that if A is a C^*-algebra any algebra homomorphism $\varphi : A \to \mathbb{C}$ is automatically $*$-preserving. Namely, for any $a \in A$,

$$\varphi(a^*) = \widehat{a^*}(\varphi) = \overline{\widehat{a}(\varphi)} = \overline{\varphi(a)}.$$

Theorem 1.29 (Spectral permanence)**.** Let A be a unital C^*-algebra and let B be a closed subalgebra containing the unit. Then for any $a \in B$, we have $\sigma_A(a) = \sigma_B(a)$.

Proof. Clearly, $\sigma_A(a) \subseteq \sigma_B(a)$. Indeed, if $\lambda - a$ is not invertible in A, then it cannot be invertible in B. To show the converse, it suffices to show that if $a \in B$ is invertible in A, then $a^{-1} \in B$ (i.e. a is even invertible in B).
Suppose first that a is self-adjoint and let $B_0 \subseteq A$ be the unital C^*-algebra generated by a and a^{-1} and let $B_0' \subseteq B$ be the unital C^*-algebra generated by a. We want to show that $B_0 \subseteq B$, and we will do this by establishing that $B_0 = B_0'$. To this end, notice that both B_0 and B_0' are commutative since a is self-adjoint. Hence by Thm. 1.27, $B_0 \cong C(\Gamma_{B_0})$ is generated by the functions $\widehat{a}$ and $\widehat{a}^{-1}$ and the subalgebra $\widehat{B_0'} \subseteq C(\Gamma_{B_0})$ is generated by the function $\widehat{a}$. First observe that for all $\varphi \in \Gamma_{B_0}$, we have $1 = \varphi(\mathbf{1}) = \varphi(a)\varphi(a^{-1})$, hence $\varphi(a^{-1}) = 1/\varphi(a)$. We apply the theorem of Stone-Weierstraß. To this end, we have to show

(i) $\widehat{B_0'}$ separates points: For $\varphi, \psi \in \Gamma_{B_0}$, suppose that $f(\varphi) = f(\psi)$ for all $f \in \widehat{B_0'}$. Then in particular for $f = \widehat{a}$, i.e. $\varphi(a) = \psi(a)$. But then also $\varphi(a^{-1}) = 1/\varphi(a) = 1/\psi(a) = \psi(a^{-1})$, hence φ and ψ agree on all Laurent polynomials in a. Since those are dense in B_0, we must have $\varphi = \psi$.

(ii) No evaluation functional vanishes: Suppose that there exists $\varphi \in \Gamma_{B_0}$ such that $f(\varphi) = 0$ for all $f \in \widehat{B_0'}$. Then in particular $\varphi(a) = 0$, a contradiction to $1 = \varphi(a)\varphi(a^{-1})$.

(iii) $\widehat{B_0'}$ is closed with respect to complex conjugation: This is clear, since B_0' is a $*$-algebra and the Gelfand transform is $*$-preserving.

We conclude that $\widehat{B_0'}$ is dense in $C(\Gamma_{B_0})$. But it is also closed, hence $\widehat{B_0'} = C(\Gamma_{B_0})$ and $B_0' = B_0$, which was to show.
Finally, let $a \in B$ be invertible but not necessarily self-adjoint. Then $(a^{-1})^*$ is an inverse for a^*, hence the self-adjoint element $a^*a \in B$ is invertible in A with inverse $(a^*a)^{-1} = a^{-1}(a^*)^{-1}$. By the previous step, $(a^*a)^{-1} \in B$, hence also $a^{-1} = (a^*a)^{-1}a^* \in B$. □

Theorem 1.30 (Continuous functional calculus)**.** Let A be a unital C^*-algebra and let $a \in A$ be normal. Then there exists an isometric $*$-homomorphism $C(\sigma(a)) \to A$, $f \mapsto f(a)$, such that the identity function on $\sigma(a)$ is mapped to a.

Proof. Let $A_0 \subset A$ be the C^*-algebra generated by a, in other words the closure of the subalgebra of all polynomials in a and a^*. Since a is normal, A_0 is commutative, hence Gelfand transform gives an isomorphism $A_0 \cong C(\Gamma_{A_0})$.
We claim that $\widehat{a} : \Gamma_{A_0} \to \{\widehat{a}(\varphi) \mid \varphi \in \Gamma_{A_0}\}$ is a homeomorphism (here we used Example 1.12 and Prop. 1.25(d)). Clearly $\widehat{a}$ is surjective. We claim that $\widehat{a}$ is injective. Suppose that $\widehat{a}(\varphi) = \widehat{a}(\psi)$, i.e. $\varphi(a) = \psi(a)$. Then also

$$\varphi(a^*) = \widehat{a}(\varphi^*) = \overline{\widehat{a}(\varphi)} = \overline{\widehat{a}(\psi)} = \widehat{a}(\psi^*).$$

Since φ and ψ are multiplicative, φ and ψ agree on finite sums of $a^n(a^*)^m$. Since φ is continuous, $\varphi = \psi$ everywhere so that $\widehat{a}$ is injective. Now $\widehat{a}$ is a bijective continuous map between two Hausdorff spaces. We have to show that the inverse $f = \widehat{a}^{-1}$ is continuous. To this end, let $K \subset \Gamma_{A_0}$ be closed. Then the preimage of K under f is $f^{-1}(K) = \widehat{a}(K)$. Because Γ_{A_0} is compact, so is K. Since $\widehat{a}$ is continuous, $\widehat{a}(K)$ is compact, hence closed. This shows that the preimages of closed sets under f are closed, so f is continuous.
Finally, proof is finished by the calculation

$$\begin{aligned}
\{\widehat{a}(\varphi) \mid \varphi \in \Gamma_{A_0}\} &= \sigma(\widehat{a}) && \text{(Example 1.12)} \\
&= \sigma_{A_0}(a) && \text{(Prop. 1.25(d))} \\
&= \sigma_A(a). && \text{(Thm. 1.29).}
\end{aligned}$$
□

Using the continuous functional calculus, one can show the following result on *polar decomposition* in a unital C^*-algebra A.

Corollary 1.31 (Polar decomposition)**.** Let A be a unital C^*-algebra and let $a \in A$ be invertible. Then there exists a unitary $u \in A$ such that $a = u|a|$.

Here a^*a is self-adjoint and has non-negative spectrum by Prop. 1.14(f), hence its square-root $|a| = (a^*a)^{1/2}$ can be defined using Thm. 1.30.

Proof. Clearly, u is given by $u = a|a|^{-1}$. We have to show that it is unitary. To this end, we calculate

$$u^*u = (a^*a)^{-1/2}a^*a(a^*a)^{-1/2} = (a^*a)^{-1/2}(a^*a)^{-1/2}a^*a = \mathbf{1},$$

using that the map $C(\sigma(a^*a)) \to A$ is an algebra homomorphism. Showing that $uu^* = \mathbf{1}$ is more involved.
We first claim that $\sigma(a^*a) = \sigma(aa^*)$. To this end, let $\lambda \in \rho(a^*a)$, in other words $\lambda - a^*a$ is invertible. Since a is invertible, this is equivalent to the invertibility of $(\lambda - a^*a)a^* = a^*(\lambda - aa^*)$, which is then equivalent to $\lambda - aa^*$ being invertible. Hence $\rho(a^*a) = \rho(aa^*)$, which implies the result on the spectra.
We now claim that $f(a^*a)a^* = a^*f(aa^*)$ for all $f \in C(\sigma(a^*a)) = C(\sigma(aa^*))$. Because of the calculation

$$(a^*a)^k a^* = (a^*a)(a^*a)\cdots(a^*a)a^* = a^*(aa^*)(a\cdots a^*)(aa^*) = a^*(aa^*)^k,$$

this is true for any polynomial f. Since polynomials are dense in $C(\sigma(a^*a))$ by the Weistraß approximation theorem, the claim follows.
Finally, we have

$$uu^* = a(a^*a)^{-1/2}(a^*a)^{-1/2}a^* = a(a^*a)^{-1}a^* = aa^*(aa^*)^{-1} = \mathbf{1},$$

where in the second step, we used the identity $f(a^*a)a^* = a^*f(aa^*)$ with $f(x) = x^{-1/2}$. □

Corollary 1.32. Let A, B be unital C^*-algebras and let $\Phi : A \to B$ be an injective unital $*$-homomorphism. Then Φ is isometric.

Proof. Assume the converse. Since by Prop. 1.16, Φ is contractive, there this would mean that there exists $a \in A$ with $\|a\| = 1$, but $\|\Phi(a)\| =: \lambda_0 < 1$. By the C^*-identity, also $\|a^*a\| = 1$ and Since Φ is a $*$-homomorphism, also $\|\Phi(a^*a)\| = \lambda_0^2 < 1$. Therefore, we can choose a continuous function $f \in C(\sigma(a^*a))$ such that $f(\lambda) = 0$ on $[0, \lambda_0^2]$ and $f(1) = 1$. As seen in the proof of Prop. 1.16, $\sigma(\Phi(a^*a)) \subset \sigma(a^*a)$. We claim that $\Phi(f(a^*a)) = f(\Phi(a^*a))$. Indeed, this holds for f a polynomial since Φ is a homomorphism; the general case follows since polynomials are dense in $C(\sigma(a^*a))$. Now $\|a^*a\| = 1$ means that $1 \in \sigma(a^*a)$ (Prop. 1.14(d)), hence

$$\|f(a^*a)\| = \sup\{|f(\lambda)\| \mid \lambda \in \sigma(a^*a)\} = 1.$$

But since $\|\Phi(a^*a)\| = \lambda_0^2$, we have $\sigma(\Phi(a^*a)) \subset [0, \lambda_0^2]$, hence $\Phi(f(a^*a)) = f(\Phi(a^*a)) = 0$. This contradicts the injectivity of Φ. $\square$

Definition 1.33 (Representation)**.** Let A be a C^*-algebra.

(a) If H is a Hilbert space, a $*$-homomorphism $\rho : A \to \mathbb{B}(H)$ is called a *$*$-representation*.

(b) A $*$-representation is called *faithful* if it is injective.

(c) A $*$-representation on a Hilbert space H is called *irreducible* if whenever $V \subseteq H$ is a closed subspace such that $\rho(a)v = 0$ for all $a \in A$, $v \in H$, then $V = \{0\}$.

Theorem 1.34 (Gelfand-Naimark, non-commutative version)**.** Let A be a C^*-algebra. Then there exists a Hilbert space H together with a faithful and isometric $*$-representation $\rho : A \to \mathbb{B}(H)$.

Proof sketch. After possibly replacing A by A^+, we may assume that A is unital. The Gelfand space Γ_A is "too small" to characterize A when it is not commutative; indeed, since for $\varphi \in \Gamma_A$,

$$\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0,$$

the Gelfand space Γ_A only depends on the commutator subspace $A/[A, A]$.
In the non-commutative case, we therefore instead consider the space

$$S_A := \{\varphi : A \to \mathbb{C} \text{ continuous} \mid \forall a \in A : \varphi(a^*a) \geq 0,\ \|\varphi\| = 1.\} \subset A',$$

where we give up on the requirement that φ is multiplicative. For any such φ, the obtain a (semi-) positive Hermitean form $\langle a, b\rangle := \varphi(a^*b)$ on A. The corresponding completion H_φ is a Hilbert space that comes with a $*$-representation $\rho_\varphi : A \to \mathbb{B}(H)$ defined by $\rho_\varphi(a)[b] := [ab]$. One then defines

$$H = \bigoplus_{\varphi \in S_A} H_\varphi, \qquad \rho = \bigoplus_{\varphi \in S_A} \rho_\varphi \tag{6}$$

and shows that the corresponding ρ is faithful. It is isometric by Corollary 1.32. $\square$

Remark 1.35. As obvious from formula (6), the representation of A constructed in the proof above is typically not separable (namely as soon as S_A is an uncountable set).

Some algebras in fact to not have a separable representation, e.g. the *Calkin algebra* $Q(H) := \mathbb{B}(H)/\mathbb{K}(H)$, for H a separable Hilbert space [5, Satz IX.3.16].

1.5 The spatial tensor product

In this section, we define the spatial tensor product of C^*-algebras, in particular in order to put C^*-norms on matrix algebras $M_n(A)$ over C^*-algebras A. General references for the theory of tensor products on C^*-algebras are [4, §T.5] and [1, §3].
For $*$-algebras A, B, the algebraic tensor product $A \otimes_{\text{alg}} B$ is an algebra, with product determined and well-defined (!) by $a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 a_2 \otimes b_1 b_2$ for $a_1, a_2 \in A$, $b_1, b_2 \in B$. It is moreover a $*$-algebra with the $*$-operation $(a \otimes b)^* = a^* \otimes b^*$.

Definition 1.36 (Spatial tensor product)**.** Let A and B be C^*-algebras.

(a) The *spatial norm* on $A \otimes_{\text{alg}} B$ is defined by

$$\|x\|_\sigma := \sup \|(\rho_A \otimes_{\text{alg}} \rho_B)(x)\| = \sup \Big\| \sum_{i=1}^{n} \rho_A(a_i) \otimes \rho_B(b_i) \Big\| \tag{7}$$

for $x = \sum_{n=1}^{m} a_n \otimes b_n \in A \otimes_{\text{alg}} B$, where the supremum is taken over all $*$-representations ρ_A, ρ_B of A and B on Hilbert spaces H, K.

(b) The *spatial tensor product* of A and B, denoted by $A \otimes B$, is the completion of $A \otimes_{\text{alg}} B$ with respect to the spatial norm.

Some comments on the definition of the spatial norm are in order. First, any pair of $*$-representations ρ_A, ρ_B on Hilbert spaces H, K defines a $*$-representation

$$\rho_A \otimes_{\text{alg}} \rho_B : A \otimes_{\text{alg}} B \longrightarrow \mathbb{B}(H) \otimes_{\text{alg}} \mathbb{B}(K) \subseteq \mathbb{B}(H \otimes K).$$

Here an operator $X \otimes Y \in \mathbb{B}(H) \otimes_{\text{alg}} \mathbb{B}(K)$ is viewed as operator in $\mathbb{B}(H \otimes K)$; explicitly, it is given by $(X \otimes Y)(\sum_i v_i \otimes w_i) = \sum_i X(v_i) \otimes Y(w_i)$. The norm is finite, since

$$\Big\| \sum_{i=1}^{n} \rho_A(a_i) \otimes \rho_B(b_i) \Big\| \leq \sum_{i=1}^{n} \|\rho_A(a_i)\| \|\rho_B(b_i)\| \leq \sum_{i=1}^{n} \|a_i\| \|b_i\|; \tag{8}$$

here we used that $*$-homomorphisms are contractive, by (1.16). It is non-degenerate, as any C^*-algebra has a faithful representation on a Hilbert space (by the Gelfand-Naimark theorem 1.34) and the induced representation $\rho_A \otimes_{\text{alg}} \rho_B$ is injective if ρ_A and ρ_B are (see e.g. [4, T.5.1]). Moreover, it is clear from the definition that $\|\cdot\|_\sigma$ satisfies the C^*-identity, hence $A \otimes B$ is a C^*-algebra.

Remark 1.37. Any pair of representations ρ_A, ρ_B of A and B induces a C^*-seminorm on $A \otimes_{\mathrm{alg}} B$ by pulling back the operator norm along $\rho_A \otimes_{\mathrm{alg}} \rho_B$. Now if ρ'_A, ρ'_B is any other pair of representations, then the direct sum representation $\rho_A \oplus \rho'_A$, $\rho_B \oplus \rho'_B$ clearly induces a *larger* seminorm this way. This shows that in the supremum in (8), it suffices to only consider *faithful* representations, because $\rho_A \oplus \rho'_A$ is faithful as soon as one of ρ_A, ρ'_A is faithful. It is also easy to see that we may restrict attention to irreducible representations.

Lemma 1.38. For any C^*-algebra A, we have $A \otimes M_n(\mathbb{C}) \cong M_n(A)$.

Proof. Clearly, $M_n(A) \cong A \otimes_{\mathrm{alg}} M_n(\mathbb{C})$, so the point is to show that $A \otimes_{\mathrm{alg}} M_n(\mathbb{C})$ is already complete with respect to the spatial norm.
Let ρ be a faithful $*$-representation of A. The representation $\rho_n : M_n(A) \to \mathbb{B}(H^n)$ induced by ρ as in (3) takes the form

$$\rho_n\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \Phi(a_{11})v_1 + \cdots + \Phi(a_{1n})v_n \\ \vdots \\ \Phi(a_{n1})v_1 + \cdots + \Phi(a_{nn})v_n \end{pmatrix}. \tag{9}$$

We check that $\rho_n(M_n(A))$ is complete in $\mathbb{B}(H^n)$. To this end, we observe that for any $X = (X_{ij})_{1\le i,j\le n} \in \mathbb{B}(H^n)$, one has for each $i, j = 1, \dots, n$

$$\left\|\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}\right\|^2 = \sum_{i=1}^n \sup_{\|v\|=1} \big\|X_{i1}v_1 + \cdots + X_{in}v_n\big\|^2 \ge \|X_{ij}\|^2.$$

We conclude that, if a sequence $a^{(m)} = (a_{ij}^{(m)})_{1\le i,j\le n} \in M_n(A)$, $m \in \mathbb{N}$, is is such that $\rho_n(a^{(m)}) \to X \in \mathbb{B}(H^n)$, then also $\rho(a_{ij}^{(m)}) \to X_{ij}$ for all $1 \le i, j \le n$. Since the image of ρ is closed, this implies $X_{ij} = \rho(a_{ij})$ for some $a_{ij} \in A$, hence X is in the image of ρ_n.
Now since the image of ρ_n is closed, pulling back the norm on $\mathbb{B}(H^n)$ to $M_n(A)$ via ρ_n gives a complete norm on $M_n(A)$ satisfying the C^*-identity. We have to show that this norm coincides with the spatial norm. By Remark 1.37, it suffices to consider faithful, irreducible representations in the definition of the spatial norm; for $M_n(\mathbb{C})$, any such representation is isomorphic to the standard representation $\mathrm{id}_{M_n(\mathbb{C})} : M_n(\mathbb{C}) \to \mathbb{B}(\mathbb{C}^n) = M_n(\mathbb{C})$. Also, under the isomorphism $M_n(A) \cong A \otimes M_n(\mathbb{C})$, we have $\rho_n = \rho \otimes_{\mathrm{alg}} \mathrm{id}_{M_n(\mathbb{C})}$. Combining these two observations, we conclude that the spatial norm of $a \in M_n(A) \cong A \otimes_{\mathrm{alg}} M_n(\mathbb{C})$ is given by

$$\|a\|_\sigma = \sup_\rho \|\rho_n(a)\|,$$

where the supremum is taken over all faithful representations ρ of A. On the other hand, we have seen above that each of the norms $\|a\|_\rho := \|\rho_n(a)\|$ turns $M_n(A)$ into a C^*-algebra, hence they must all be equal, by Prop. 1.15. This finishes the proof. $\square$

Theorem 1.39. Let A and B be C^*-algebras. Then the supremum (7) is in fact a maximum, which is taken at any pair of *faithful* representations ρ_A, ρ_B. In other words, for any $x \in A \otimes_{\mathrm{alg}} B$ and any pair of faithful representations ρ_A, ρ_B of A, B, we have

$$\|x\|_\sigma = \|(\rho_A \otimes_{\mathrm{alg}} \rho_B)(x)\|.$$

In particular, if $A \subseteq \mathbb{B}(H)$, $B \subseteq \mathbb{B}(K)$ are C^*-subalgebras, then $A \otimes B$ is isomorphic to the norm closure of $A \otimes_{\mathrm{alg}} B \subseteq \mathbb{B}(H \otimes K)$.

Proof. We first observe that by Lemma 1.38, the theorem is true if A is isomorphic to $M_n(\mathbb{C})$. In particular, we have

$$\|x\|_\sigma = \|(\mathrm{id}_A \otimes \rho_B)(x)\|$$

for any faithful representation ρ_B of B. Namely, for any such representation, the right hand side gives a complete C^*-norm on $M_n(\mathbb{C}) \otimes B$, which then must be all equal.
In general, let ρ_A and ρ_B representations of A and B and let ρ'_B be faithful representation of B. We will show that for all $x \in A \otimes_{\mathrm{alg}} B$, we have

$$\|(\rho_A \otimes_{\mathrm{alg}} \rho_B)(x)\| \leq \|(\rho_A \otimes_{\mathrm{alg}} \rho'_B)(x)\|. \tag{10}$$

By symmetry of the tensor product construction, the same is true when replacing ρ_A by a faithful representation $\rho_{A'}$, and the proposition follows.
To begin with, let $\mathcal{V}$ be the directed system of finite-dimensional subspaces of H (see Example 2.26). For $V \in \mathcal{V}$, let P_V be the orthogonal projection onto V in H, and let $P'_V := P_V \otimes \mathrm{id}_K$, the orthogonal projection onto $V \otimes K$ in $H \otimes K$. It is then an easy lemma from functional analysis that

$$\|X\| = \lim_{\mathcal{V}} \|(P_V \otimes \mathrm{id}_{\mathbb{B}(K)})X(P_V \otimes \mathrm{id}_{\mathbb{B}(K)})\|,$$

for all $X \in \mathbb{B}(H \otimes K)$, where the limit is taken in the sense of nets.
We obtain that for any $x \in A \otimes_{\mathrm{alg}} B$,

$$\|(\rho_A \otimes_{\mathrm{alg}} \rho_B)(x)\| = \lim_{\mathcal{V}} \|(P_V \rho_A P_V \otimes_{\mathrm{alg}} \rho_B)(x)\|.$$

For each $V \in \mathcal{V}$, the map $P_V\rho_A P_V \otimes_{\mathrm{alg}} \rho_B$ is the composition of the linear map $P_V \rho_A P_V \otimes \mathrm{id}_B : A \otimes_{\mathrm{alg}} B \to \mathbb{B}(V) \otimes_{\mathrm{alg}} B$ and the $*$-homomorphism $\mathrm{id}_{\mathbb{B}(V)} \otimes_{\mathrm{alg}} \rho_B : \mathbb{B}(V) \otimes_{\mathrm{alg}} B \to \mathbb{B}(V \otimes K)$. Therefore, for any $x \in A \otimes_{\mathrm{alg}} B$, we have

$$\|(\rho_A \otimes_{\mathrm{alg}} \rho_B)(x)\| = \lim_{\mathcal{V}} \|(\mathrm{id}_{\mathbb{B}(V)} \otimes \rho_B)(P_V \rho_A P_V \otimes \mathrm{id}_B)(x)\|.$$

Now since $\mathrm{id}_{\mathbb{B}(V)} \otimes \rho_B$ is a $*$-homomorphism, hence contractive (Prop. 1.16), we have

$$\|(\mathrm{id}_{\mathbb{B}(V)} \otimes \rho_B)(P_V \rho_A P_V \otimes \mathrm{id}_B)(x)\| \leq \|(P_V \rho_A P_V \otimes \mathrm{id}_B)(x)\|_\sigma, \tag{11}$$

where the right hand side denotes the spatial norm of $\mathbb{B}(V) \otimes_{\mathrm{alg}} B = \mathbb{B}(V) \otimes B$. Moreover, by (the proof of) Lemma 1.38, we have equality in (11) if ρ'_B is faithful. Combining these observations, we get that for ρ_B an arbitrary $*$-representation and ρ'_B a faithful $*$-representation, we have

$$\|P_V \rho_A P_V \otimes_{\mathrm{alg}} \rho_B)(x)\| \leq \|(P_V \rho_A P_V \otimes_{\mathrm{alg}} \rho'_B)(x)\|$$

for all $V \in \mathcal{V}$. Taking the limit over $\mathcal{V}$, we obtain (10), which finishes the proof. $\square$

If $\Phi : A \to A'$ and $\Psi : B \to B'$ are $*$-homomorphisms, we get an induced $*$-homomorphism $\Phi \otimes_{\mathrm{alg}} \Psi : A \otimes_{\mathrm{alg}} B \to A' \otimes_{\mathrm{alg}} B'$. It is continuous because for $*$-representations $\rho_{A'}$ and $\rho_{B'}$ of A' and B', $\rho_{A'} \circ \Phi$ and $\rho_{B'} \circ \Psi$ are $*$-representations of A, respectively B. Therefore $\Phi \otimes_{\mathrm{alg}} \Psi$ extends by continuity to a $*$-homomorphism $\Phi \otimes \Psi : A \otimes B \to A' \otimes B'$. We record the following consequence of Thm. 1.39 for later use.

Corollary 1.40. Let A, A', B and B' be C^*-algebras and let $\Phi : A \to A'$, $\Psi : B \to B'$ be injective $*$-homomorphisms. Then $\Phi \otimes \Psi : A \otimes B \to A' \otimes B'$ is injective.

Proof. If $\rho_{A'}$ and $\rho_{B'}$ are faithful representations of A', respectively B', then $\rho_A := \rho_{A'} \circ \Phi$ and $\rho_B := \rho_{B'} \circ \Psi$ are faithful representations of A, respectively B. Hence for all $x \in A \otimes_{\mathrm{alg}} B$,

$$\|(\Phi \otimes_{\mathrm{alg}} \Psi)(x)\|_\sigma = \|(\rho_{A'} \otimes_{\mathrm{alg}} \rho_{B'})(\Phi \otimes_{\mathrm{alg}} \Psi)(x)\| = \|(\rho_A \otimes \rho_B)(x)\| = \|x\|.$$

This shows that $\Phi \otimes \Psi$ is isometric, in particular injective. $\square$

Example 1.41. Let A be a C^*-algebra and let X be a compact Hausdorff space. Then $C_0(X) \otimes A \cong C_0(X, A)$, the C^*-algebra of continuous A-valued functions on X vanishing at infinity.
To see this, observe first there is an obvious injective $*$-homomorphism $\Phi : C_0(X) \otimes_{\mathrm{alg}} A \to C_0(X, A)$, given by $\Phi(f \otimes a)(t) = f(t)a$. To see that its image is dense, one first observes that $C_c(X, A)$ (compactly supported functions) is dense in $C_0(X, A)$, hence it suffices to approximate a given compactly supported function f. This is done using a partition of unity subordinate to a suitable finite open cover of the support of f. For details, see for example [4, §T.2, p. 322].
On the other hand, we claim that Φ is continuous with respect to the spatial norm on $C(X) \otimes_{\mathrm{alg}} A$. To this end, for a Hilbert space H, let $\ell^2(X, H)$ be Hilbert space of functions $\alpha : X \to \mathbb{C}$ with countable support $x_1, x_2, \dots \in X$ such that $\sum_{n=1}^\infty |\alpha(x_n)|^2 < \infty$; we write $\ell^2(X)$ if $H = \mathbb{C}$. Now we have a faithful $*$-representation $\mu : C(X) \to \ell^2(X)$, given by $\mu(f)\alpha = f \cdot \alpha$, and given a faithful representation ρ_A of A on a Hilbert space H, the $*$-representation $\theta \otimes_{\mathrm{alg}} \rho_A$ of $C(X) \otimes_{\mathrm{alg}} A$ on $\ell^2(X) \otimes H = \ell^2(X, H)$ takes has an obvious extension to an injective (hence isometric) $*$-representation ρ of $C(X, A)$ on $\ell^2(X, H)$

such that $\rho \circ \Phi = \theta \otimes_{\mathrm{alg}} \rho_A$. Now for any $x \in C(X) \otimes_{\mathrm{alg}} A$, we have

$$\|\Phi(x)\| = \|(\rho \circ \Phi)(x)\| = \|(\theta \otimes_{\mathrm{alg}} \rho_A)(x)\| = \|x\|_\sigma.$$

Here in the last step, we used Thm. 1.39. This finishes the proof.

2 The K_0-functor

In this section, we define the K_0-group $K_0(A)$ of a C^*-algebra A. We then state and prove its main properties.

2.1 Equivalence of projections

Throughout this section, for C^*-algebras A, we denote by $\widetilde{A}$ the C^*-algebra given by A if A is unital and A^+ if A is non-unital.

Definition 2.1 (Projections, Isometries, Unitaries)**.** Let A be a C^*-algebra.

(a) $p \in A$ is called *projection* if $p^* = p$ and $p^2 = p$.

(b) $v \in A$ is called *partial isometry* if v^*v is a projection.

(c) If A is unital, $u \in A$ is called *unitary* if $u^*u = uu^* = \mathbf{1}$.

Lemma 2.2. Let A be a C^*-algebra and let $v \in A$ be a partial isometry, so that $p = v^*v$ is a projection. Then also $q = vv^*$ is a projection. Moreover,

$$v = vv^*v = vp = qv, \qquad v^* = v^*vv^* = pv^* = v^*q. \tag{12}$$

Proof. By the C^*-identity, we have

$$\|v - vv^*v\|^2 = \|v(\mathbf{1} - v^*v)\|^2 = \|(\mathbf{1} - v^*v)v^*v(\mathbf{1} - v^*v)\| = \|(\mathbf{1} - p)p(\mathbf{1} - p)\| = 0,$$

hence $v = vv^*v$. Taking the transpose shows $v^* = v^*vv^*$, so that we have established (12). q is a projection because it is self-adjoint and $q^2 = vv^*vv^* = vv^* = q$, using (12). $\square$

Definition 2.3 (Equivalence of projections)**.** Let A be a C^*-algebra and let $p, q \in A$ be projections.

(a) p, q are called *Murray-von-Neumann equivalent*, denoted $p \sim q$, if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(b) p, q are called *unitarily equivalent*, denoted $p \sim_u q$, if there exists a unitary $u \in \widetilde{A}$ such that $q = upu^*$.

(c) p, q are called *homotopy equivalent*, denoted $p \sim_h q$, if there exists a continuous path $(p_t)_{t\in[0,1]}$ of projections in A, such that $p_0 = p$, $p_1 = q$. Such a path is called *homotopy between* p and q.

Lemma 2.4. All the relations in Def. 2.3 are equivalence relations on the set of projections in A.

Proof. The only non-trivial part is the transitivity of Murray-von-Neumann equivalence. Let $p, q, r \in A$ be three projections, and let $v, w \in A$ be partial isometries with $v^*v = p$, $vv^* = q$, $w^*w = q$, $ww^* = r$. Then

$$(wv)^*(wv) = v^*w^*wv = v^*qv = v^*vv^*v = p^2 = p,$$
$$(wv)(wv)^* = wvv^*w = wqw^* = ww^*ww^* = r^2 = r.$$

□

Lemma 2.5. Let A be a C^*-algebra and let p and q be projections in A with $\|p - q\| < 1/2$. Then there exists a unitary $u \in \widetilde{A}$ with $q = upu^*$.

Proof. Set $a = qp + (\mathbf{1} - q)(\mathbf{1} - p) \in \widetilde{A}$. It satisfies

$$qa = qp = ap. \tag{13}$$

Because

$$\|\mathbf{1} - a\| = \|q + p - 2qp\| = \|(\mathbf{1} - q)(p - q) - q(p - q)\| \leq 2\|p - q\| < 1,$$

the element a is invertible, with $a^{-1} = \sum_{n=0}^{\infty}(\mathbf{1} - a)^n$ (Neumann series), and from (13) follows $q = apa^{-1}$.
To obtain a unitary u with $q = upu^*$, we take the unitary $u = a|a|^{-1}$ from the polar decomposition of a, see Corollary 1.31 (remember that $|a| = (a^*a)^{1/2}$).
We claim that p commutes with $|a|^{-1}$. First, p commutes with $|a|^2$ because of the

calculation

$$|a|^2p = a^*ap = a^*qa = (qa)^*a = (ap)^*a = pa^*a = p|a|^2,$$

where we used (13). We conclude that it commutes with all elements of the C*-subalgebra $B \subseteq A$ generated by $|a|^2$ and $\mathbf{1}$. Since $|a|^2$ is invertible, B contains $|a|^{-1}$, proving the claim.
Using (13) again, we calculate

$$upu^* = a|a|^{-1}pu^* = ap|a|^{-1}u^* = qa|a|^{-1}u^* = quu^* = q.$$

□

Proposition 2.6 (Equivalence of equivalences)**.** Let A be a C*-algebra and let p, q be projections in A. Then

(a) $p \sim_h q \Longrightarrow p \sim_u q$;

(b) $p \sim_u q \Longrightarrow p \sim q$;

(c) $p \sim q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$;

(d) $p \sim_u q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.

Proof. (a) Let $(p_t)_{t\in[0,1]}$ be a homotopy between p and q and choose a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|p_{t_i} - p_{t_{i-1}}\| < 1/2$ for all $i = 1, \ldots, n$. By Lemma 2.5, there exist unitaries u_i in $\widetilde{A}$ such that $p_{t_i} = u_i p_{t_{i-1}} u_i^*$ for each i, hence with $u = u_n \cdots u_1$, we have $p_1 = up_0u^*$.
(b) If $q = upu^*$ for some unitary $u \in \widetilde{A}$, then with $v = up$, we have $v^*v = pu^*up = p^2 = p$ and $vv^* = up^2u^* = upu^* = q$.
(c) Let $v \in A$ be a partial isometry with $v^*v = p$, $vv^* = q$. We need to find an element $u \in \widetilde{M_2(A)}$ with $u\,\mathrm{diag}(p, 0)u^* = \mathrm{diag}(q, 0)$. To this end, define elements of $M_2(\widetilde{A})$ by

$$w := \begin{pmatrix} v & \mathbf{1} - q \\ \mathbf{1} - p & v^* \end{pmatrix}, \qquad s := \begin{pmatrix} q & \mathbf{1} - q \\ \mathbf{1} - q & q \end{pmatrix}.$$

Clearly, s is unitary. Moroever,

$$w^*w = \begin{pmatrix} v^*v + (\mathbf{1} - p)^2 & v^* - v^*q + v^* - pv^* \\ v - qv + v - vp & (\mathbf{1} - q)^2 + vv^* \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

where we used the identities (12). Similarly, one calculates $ww^* = \mathrm{diag}(\mathbf{1}, \mathbf{1})$, hence

u is unitary. Also

$$w \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w^* = w \begin{pmatrix} pv^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} vpv^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} vv^*q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix},$$

again using (12). However, if A is non-unital, we generally have $w \notin \widetilde{M_2(A)}$. To repair this, set $u = sw$ and notice that also $u\,\mathrm{diag}(p, 0)u^* = \mathrm{diag}(q, 0)$ and since $A \subseteq \widetilde{A}$ is an ideal,

$$u = sw = \begin{pmatrix} qv + (\mathbf{1} - q)(\mathbf{1} - p) & (\mathbf{1} - q)v^* \\ (\mathbf{1} - q)v + q(\mathbf{1} - p) & \mathbf{1} - q + qv^* \end{pmatrix} \in \widetilde{M_2(A)},$$

(d) Let $u \in \widetilde{A}$ be a unitary such that $q = upu^*$. For $t \in [0, 1]$, define elements of $M_2(\widetilde{A})$ by

$$r_t := \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}, \qquad w_t := \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix} r_t \begin{pmatrix} u^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} r_t^*, \qquad p_t := w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*.$$

Then $(w_t)_{t\in[0,1]}$ is a path of unitaries in $M_2(\widetilde{A})$ with $w_0 = \mathrm{diag}(\mathbf{1}, \mathbf{1})$, $w_1 = \mathrm{diag}(u, u^*)$ and $(p_t)_{t\in[0,1]}$ is a path of projections in $M_2(A)$ with $p_0 = \mathrm{diag}(p, 0)$ and $p_1 = \mathrm{diag}(q, 0)$. □

2.2 The Grothendieck construction

Remember that a *semigroup* is a non-empty set S together with a map $S \times S \to S$, $(g, h) \mapsto g \cdot h$, that satisfies $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all $g, h, k \in S$ (*associativity*). A semigroup S is *abelian* if the multiplication is commutative, $g \cdot h = h \cdot g$ for all $g, h \in S$. A *homomorphism* between semigroups S, T is a map $\varphi : S \to T$ such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in S$. Any group is in particular a semigroup.

Definition 2.7 (Group completion)**.** Let S be an abelian semigroup. Then a *group completion* or *Grothendieck group* is an abelian group $G(S)$, together with a homomorphism $\iota_S : S \to G(S)$ satisfying the following universal property: For every abelian group H together with a homomorphism $\rho : S \to H$, there is a unique homomorphism $\widetilde{\rho} : G(S) \to H$ such that

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[dr, "\rho"'] & G(S) \arrow[d, dashed, "\exists!"] \\ & H \end{tikzcd} \qquad \text{commutes.}$$

Proposition 2.8 (Grothendieck construction)**.** Let S be an abelian semigroup. Then a group completion exists and is unique up to unique isomorphism. In fact, there exists a functor G from abelian semigroups to abelian groups such that for every semigroup S, $G(S)$ is a group completion of S.

Proof sketch. A concrete model is $G(S) := S \times S/\approx$, where the equivalence relation $\approx$ is defined by $(g, h) \approx (g', h')$ if there exists $k \in S$ such that $g + h' + k = g' + h + k$ (the additional k is needed to show transitivity of the relation, in case that S does not have the cancellation property $g+k = h+k \Rightarrow g = h$). One then shows that $G(S)$ is a group and sets $\iota_S(g) = [g + k, k]$ for any $k \in S$ (any choice of k yields the same element). To show the universal property, observe that for any $g, h \in S$, $[g, h] = \iota_S(g) - \iota_S(h)$, hence for a given homomorphism $\rho : S \to H$, we must have $\widetilde{\rho}([g, h]) = \rho(g) - \rho(h)$; this indeed gives a well-defined group homomorphism $\widetilde{\rho} : G(S) \to H$.
If now $\varphi : S \to T$ is a homomorphism of semigroups, then $\rho := \iota_T \circ \varphi$ is homomorphism S to the group $G(T)$, so by the universal property, there exists a unique group homomorphism $G(\varphi) := \widetilde{\rho} : G(S) \to G(T)$. One then verifies that for a second homomorphism $\psi : T \to U$, one has the functoriality $G(\psi) \circ G(\varphi) = G(\psi \circ \varphi)$. □

Remark 2.9. By the above, elements of $G(S)$ can be represented by equivalence classes of pairs of elements in S. We suggestively write $g - h := [g, h]$ for $g, h \in S$.

Remark 2.10. By the universal property of the Grothendieck construction, for any abelian semigroup S and any abelian group H, the canonical map

$$\mathrm{Hom}_{\mathsf{Ab}}(G(S), H) \longrightarrow \mathrm{Hom}_{\mathsf{SAb}}(S, H), \qquad \varphi \longmapsto \varphi \circ \iota_S$$

is a bijection. Since it is natural in both S and H, this shows that the functor $G : \mathsf{SAb} \to \mathsf{Ab}$ is left adjoint to the *forgetful functor* $\mathsf{Ab} \to \mathsf{SAb}$.

Example 2.11. We have $G(\mathbb{N}) = \mathbb{Z}$; in fact the left hand side can be taken as a definition of $\mathbb{Z}$.

Example 2.12. If we set $n + \infty = \infty + n = \infty$ for $n \in \mathbb{N}$ and $\infty + \infty = \infty$, then $\mathbb{N} \cup \{\infty\}$ is a semigroup. Moreover, $G(\mathbb{N} \cup \{\infty\}) = \{0\}$, because in groups, the cancellation rule holds, that is, $[n] + [\infty] = [\infty]$ implies $[n] = [0]$ for all $n \in \mathbb{N} \cup \{\infty\}$.

2.3 Definition of K_0

We have canonical embeddings $M_n(A) \to M_{n+1}(A)$ given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{14}$$

We denote by $M_\infty(A)$ the union of all the $M_n(A)$, that is, the direct limit in the category of $*$-algebra. $M_\infty(A)$ can be described as the $*$-algebra of infinite matrices with entries in A, with only finitely many non-zero entries. Since all the inclusions $M_n(A) \to M_{n+1}(A)$ are isometric, $M_\infty(A)$ inherits a norm which satisfies the C^*-property but is not complete.

Definition 2.13. Let A be a C^*-algebra.

(a) Two projections $p, q \in M_\infty(A)$ are *equivalent*, if for some $n \in \mathbb{N}$, $p, q \in M_n(A)$ and $p \sim q$ in $M_n(A)$. The set of equivalence classes is denoted by $\mathcal{V}(A)$.

(b) If $p, q \in M_\infty(A)$ are projections with $p \in M_n(A)$, $q \in M_m(A)$, we define

$$[p] + [q] := [\operatorname{diag}(p, q)], \qquad \text{where } \operatorname{diag}(p, q) \in M_{n+m}(A) \subset M_\infty(A).$$

To show well-definedness of the addition, notice that for all $p \in M_n(A)$, $q \in M_m(A)$,

$$\begin{pmatrix} p & & & & \\ & q & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} p & & & & \\ & 0 & & & \\ & & q & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \qquad \text{in } M_\infty(A).$$

Remark 2.14. By Prop. 2.6, we can use any of the equivalence relations $\sim$, $\sim_u$, $\sim_h$ to obtain the same set $\mathcal{V}(A)$.

Lemma 2.15. Let A be a C^*-algebra. Then $\mathcal{V}(A)$ is an abelian semigroup.

Proof. Let $p, q, r \in M_\infty(A)$ be projections with $p \in M_n(A)$, $q \in M_m(A)$ and $r \in M_l(A)$. Because of $\operatorname{diag}(\operatorname{diag}(p, q), r) = \operatorname{diag}(p, q, r) = \operatorname{diag}(p, \operatorname{diag}(q, r))$ in

$M_{n+m+l}(A)$, the semigroup operation is associative. With

$$v = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \qquad \text{we have} \qquad v^*v = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad vv^* = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix},$$

hence v is a partial isometry and $\mathrm{diag}(p, q) \sim \mathrm{diag}(q, p)$ in $M_{n+m}(A)$. This shows that the semigroup operation is commutative. □

If A and B are C^*-algebras and $\Phi : A \to B$ a $*$-homomorphism, one easily checks that

$$\mathcal{V}(\Phi) : \mathcal{V}(A) \to \mathcal{V}(B), \qquad [p] \mapsto [\Phi(p)]$$

gives a well-defined homomorphism of semigroups. If C is another C^*-algebra and $\Psi : B \to C$ is a $*$-homomorphism, one clearly has $\mathcal{V}(\Psi) \circ \mathcal{V}(\Phi) = \mathcal{V}(\Psi \circ \Phi)$, hence $\mathcal{V}$ is a well-defined functor from C^*-algebras to abelian semigroups.

Definition 2.16 (The K_0-functor)**.** Let A be a C^*-algebra.

(a) The group $K_0(A)$ associated to A is defined by

$$K_0(A) := \ker\big(G\mathcal{V}(\varepsilon_A) : G\mathcal{V}(A^+) \to G\mathcal{V}(\mathbb{C})\big).$$

(b) Given another C^*-algebra B with a $*$-homomorphism $\Phi : A \to B$, define $K_0(\Phi) : K_0(A) \to K_0(B)$ as the unique group homomorphism fitting in the commutative diagram

$$\begin{tikzcd} K_0(A) \arrow[r, hook] \arrow[d, "K_0(\Phi)"'] & G\mathcal{V}(A^+) \arrow[r, "G\mathcal{V}(\varepsilon_A)"] \arrow[d, "G\mathcal{V}(\Phi^+)"'] & G\mathcal{V}(\mathbb{C}) \arrow[d, equal] \\ K_0(B) \arrow[r, hook] & G\mathcal{V}(B^+) \arrow[r, "G\mathcal{V}(\varepsilon_B)"'] & G\mathcal{V}(\mathbb{C}). \end{tikzcd} \tag{15}$$

Notice that for C^*-algebras A, B and $*$-homomorphisms $\Phi : A \to B$, there is indeed a unique group homomorphism $K_0(\Phi) : K_0(A) \to K_0(B)$ making (16) commute: By injectivity of the inclusions $K_0(A) \hookrightarrow G\mathcal{V}(A^+)$ and $K_0(B) \hookrightarrow G\mathcal{V}(B^+)$, $K_0(\Phi)$ must be the restriction of $G\mathcal{V}(\Phi^+)$ to $K_0(A)$, and if $x \in K_0(A)$, then

$$0 = G\mathcal{V}(\varepsilon_A)(x) = G\mathcal{V}(\varepsilon_B)G\mathcal{V}(\Phi^+)(x),$$

hence indeed $G\mathcal{V}(\Phi^+)(x) \in \ker G\mathcal{V}(\varepsilon_B) = K_0(B)$.

Remark 2.17. If A is unital, then $A^+ \cong A \oplus \mathbb{C}$ with ε_A being the projection onto the second factor. Therefore, $G\mathcal{V}(A^+) = G\mathcal{V}(A) \oplus G\mathcal{V}(\mathbb{C})$ with $G\mathcal{V}(\varepsilon_A)$ being the projection onto the second factor. If $\Phi : A \to B$ is a unital $*$-homomorphism, then

$G\mathcal{V}(\Phi^+) = G\mathcal{V}(\Phi) \oplus G\mathcal{V}(\mathrm{id}_{\mathbb{C}})$ under this identification, hence on the subcategory of unital C^*-algebras and unital $*$-homomorphisms, the functors $G\mathcal{V}$ and K_0 are naturally isomorphic. We will therefore often write K_0 instead of $G\mathcal{V}$ for unital C^*-algebras. In particular, for *any* C^*-algebra A, we have a short exact sequence

$$0 \longrightarrow K_0(A) \longrightarrow K_0(A^+) \xrightarrow{K_0(\varepsilon_A)} K_0(\mathbb{C}) \longrightarrow 0$$

Lemma 2.18. K_0 is a functor from C^*-algebras to abelian groups.

Proof. Let A, B and C be C^*-algebras and $\Phi : A \to B$, $\Psi : B \to C$ be $*$-homomorphisms. Consider the following diagram.

$$\begin{tikzcd}
K_0(A) \arrow[r, hook] \arrow[d, "K_0(\Phi)"'] \arrow[dd, bend right=60, "K_0(\Psi\circ\Phi)"'] & G\mathcal{V}(A^+) \arrow[r, "G\mathcal{V}(\varepsilon_A)"] \arrow[d, "G\mathcal{V}(\Phi^+)"] \arrow[dd, bend right=60, "G\mathcal{V}(\Psi\circ\Phi)"'] & G\mathcal{V}(\mathbb{C}) \arrow[d, equal] \\
K_0(B) \arrow[r, hook] \arrow[d, "K_0(\Phi)"'] & G\mathcal{V}(B^+) \arrow[r, "K_0(\varepsilon_B)"'] \arrow[d, "G\mathcal{V}(\Phi^+)"] & G\mathcal{V}(\mathbb{C}) \arrow[d, equal] \\
K_0(C) \arrow[r, hook] & G\mathcal{V}(C^+) \arrow[r, "K_0(\varepsilon_C)"'] & G\mathcal{V}(\mathbb{C})
\end{tikzcd} \tag{16}$$

Commutativity of the left-most triangle is equivalent to the desired equality $K_0(\Psi) \circ K_0(\Phi) = K_0(\Psi \circ \Phi)$. All squares commute by definition of the K_0 maps, while commutativity of the other triangle follows from functoriality of $G\mathcal{V}$. In total, we obtain that the entire diagram commutes. □

Proposition 2.19 (A portrait of K_0)**.** Let A be a C^*-algebra.

(a) Any element $x \in K_0(A)$ can be written in the form $x = [p] - [\mathbf{1}_n]$ for some $n \in \mathbb{N}$, with a projection $p \in M_\infty(A^+)$ and

$$\mathbf{1}_n := \begin{pmatrix} \mathbf{1} & & & \\ & \ddots & & \\ & & \mathbf{1} & \\ & & & 0 \end{pmatrix} \in M_\infty(A^+).$$

Moreover, we can arrange p such that $p - \mathbf{1}_n \in M_\infty(A)$.

(b) For projections $p, q \in M_\infty(A^+)$, $[p] - [q] = 0$ in $K_0(A)$ if and only if there exists $m \in \mathbb{N}$ such that $\mathrm{diag}(p, \mathbf{1}_m) \sim \mathrm{diag}(q, \mathbf{1}_m)$. Here $\sim$ can be replaced by $\sim_u$ or $\sim_h$.

(c) If B is another C^*-algebra and $\Phi : A \to B$ is a homomorphism, then for all projections $p, q \in M_n(A)$, we have $K_0(\Phi)([p] - [q]) = [\Phi^+(p)] - [\Phi^+(q)]$.

Proof. (a), first part. By definition of the Grothendieck group, any element $x \in K_0(A)$ can be written as $x = [p] - [q]$ with projections $p, q \in M_\infty(A^+)$ for some $n \in \mathbb{N}$. Since q is a projection, so is $\mathbf{1}_n - q$, and

$$[q] + [\mathbf{1}_n - q] = \left[\begin{pmatrix} q & 0 \\ 0 & \mathbf{1}_n - q \end{pmatrix}\right] = [\mathbf{1}_n],$$

as

$$u \begin{pmatrix} q & 0 \\ 0 & \mathbf{1}_n - q \end{pmatrix} u^* = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for the unitary} \qquad u = \begin{pmatrix} q & \mathbf{1}_n - q \\ \mathbf{1}_n - q & q \end{pmatrix}.$$

Therefore, if $p \in M_m(A^+)$ and $q \in M_n(A^+)$,

$$[p] - [q] = [p] + [\mathbf{1}_n - q] - \big([q] + [\mathbf{1}_n - q]\big) = \left[\begin{pmatrix} p & 0 \\ 0 & \mathbf{1}_n - q \end{pmatrix}\right] - [\mathbf{1}_n],$$

as claimed.
(b) Suppose that $[p] - [q] = 0$ in $K_0(A)$ for projections $p, q \in M_n(A^+)$. By the definition of the Grothendieck group, this implies that there exists a projection $r \in M_n(A^+) \subset M_\infty(A)$ such that $\operatorname{diag}(p, r) \sim \operatorname{diag}(q, r)$. We then also have $\operatorname{diag}(p, r, \mathbf{1}_n - r) \sim \operatorname{diag}(q, r, \mathbf{1}_n - r)$. But by a similar calculation to the one just above, we have $\operatorname{diag}(p, r, \mathbf{1}_n - r) \sim \operatorname{diag}(p, \mathbf{1}_n)$ and $\operatorname{diag}(q, r, \mathbf{1}_n - r) \sim \operatorname{diag}(q, \mathbf{1}_n)$. Conversely, if $\operatorname{diag}(p, r) \sim \operatorname{diag}(q, r)$, then

$$[p] - [q] = ([p] + [r]) - ([q] + [r]) = \left[\begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}\right] - \left[\begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix}\right] = 0.$$

(a), second part. Let $x = [p] - [\mathbf{1}_n] \in K_0(A)$, where $p \in M_\infty(A^+)$ is a projection. Since $x \in \ker K_0(\varepsilon_A)$, we have $[\varepsilon_A(p)] - [\mathbf{1}_n] = 0$. By (b), there exists $m \in \mathbb{N}$ with $p \in M_m(A^+)$ and a unitary $u \in M_m(\mathbb{C}) \subseteq M_m(A^+)$ such that $u\varepsilon_A(p)u^* = \mathbf{1}_n$. Then with the projection $p' = upu^*$, we still have $x = [p'] - [\mathbf{1}_n]$, but $\varepsilon_A(p') - \mathbf{1}_n = 0$, hence $p' - \mathbf{1}_n \in M_m(A)$.
(c) This follows from the universal property of the Grothendieck group, compare the proof sketch of Prop. 2.8. $\square$

Example 2.20 (K_0 of $\mathbb{C}$)**.** Two projections $p, q \in M_n(\mathbb{C})$ are equivalent if and only if they have the same rank, and the rank is additive under taking direct sums. Therefore $\mathcal{V}(\mathbb{C}) = \mathbb{N}_0$ and $K_0(\mathbb{C}) = \mathbb{Z}$ (see Example 2.11). We conclude that the map

$$\tau : K_0(\mathbb{C}) \longrightarrow \mathbb{Z}, \qquad [p] - [q] \longmapsto \operatorname{tr}(p) - \operatorname{tr}(q) \tag{17}$$

is a well-defined group isomorphism.

Example 2.21 (K_0 of $\mathbb{B}(H)$)**.** Let H be a Hilbert space and $A = \mathbb{B}(H)$. Two projections in $M_n(A) \cong \mathbb{B}(H^n)$ are equivalent if and only if they have the same rank, and the rank is additive under taking direct sums. Here the rank can be any number in $\mathbb{N}_0 \cup \{\infty\}$, and the Grothendieck group of this semigroup is zero (Example 2.12).

2.4 Homotopy invariance

Definition 2.22 (Homotopy)**.** Let A and B be C^*-algebras.

(a) A *homotopy* between $*$-homomorphisms $\Phi, \Psi : A \to B$ is a family $(\Phi_t)_{t\in[0,1]}$ of $*$-homomorphisms such that $\Phi_0 = \Phi$, $\Phi_1 = \Psi$ and such that $t \mapsto \Phi_t(a)$ is continuous for every $a \in A$.

(b) Two $*$-homomorphisms $\Phi, \Psi : A \to B$ are called *homotopic* if there exists a homotopy between them.

(c) A $*$-homomorphism $\Phi : A \to B$ is a *homotopy equivalence* if there exists a $*$-homomorphism $\Phi' : B \to A$ such that both $\Phi \circ \Phi'$ and $\Phi' \circ \Phi$ are homotopic to the identity.

Example 2.23. If X, Y are compact topological spaces, then a continuous map $\varphi : X \to Y$ induces a $*$-homomorphism $\varphi^* : C(Y) \to C(X)$, $f \mapsto \varphi^* f$ by pullback. If φ and ψ are two such maps, then a homotopy (of continuous maps) induces a homotopy of $*$-homomorphisms between φ^* and ψ^*, and if φ is a homotopy equivalence (in the sense of topology), then φ^* is a homotopy equivalence in the sense of Def. 2.22.

Theorem 2.24 (Homotopy invariance)**.** Let A and B be C^*-algebras.

(a) If $\Phi, \Psi : A \to B$ are homotopic $*$-homomorphisms, then $K_0(\Phi) = K_0(\Psi)$.

(b) If $\Phi : A \to B$ is a homotopy equivalence, then $K_0(\Phi) : K_0(A) \to K_0(B)$ is an isomorphism.

Proof. (a) Let $(\Phi_t)_{t\in[0,1]}$ be a homotopy between Φ and Ψ. Let $x = [p] - [q] \in K_0(A)$ with $p, q \in M_n(A)$. Then $(\Phi_t^+(p))_{t\in[0,1]}$ and $(\Phi_t^+(q))_{t\in[0,1]}$ are homotopies of projections between $\Phi^+(p)$ and $\Psi^+(p)$, respectively $\Phi^+(q)$, $\Psi^+(q)$. Hence by Prop. 2.19(c),

$$K_0(\Phi)(x) = [\Phi^+(p)] - [\Phi^+(q)] = [\Psi^+(p)] - [\Psi^+(q)] = K_0(\Psi)(x).$$

(b) If $\Phi : A \to B$ is a homotopy equivalence with homotopy inverse Φ', then by functoriality of K_0 and the results of (a),

$$\mathrm{id}_{K_0(A)} = K_0(\mathrm{id}_A) = K_0(\Phi') \circ K_0(\Phi) \qquad \text{and} \qquad \mathrm{id}_{K_0(B)} = K_0(\mathrm{id}_B) = K_0(\Phi) \circ K_0(\Phi').$$

Hence $K_0(\Phi)$ and $K_0(\Phi')$ must be isomorphisms. $\square$

2.5 Continuity

Definition 2.25 (Directed set)**.** A directed set is a set I with a partial order $\leq$ such that any two elements have a common upper bound. In other words, for all objects $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Example 2.26. Any subset $S \subseteq \mathbb{R}$ gives rise to a directed set with the usual order relation. The same statement is true for S replaced by any totally ordered set. An example for a directed set which does not come from a total order is the set of finite-dimensional subspaces of a Hilbert space H, ordered by inclusion.

Definition 2.27 (Direct limits)**.** Let C be a category and let I be a directed set.

(a) A *direct system* in C is a collection of objects c_i, $i \in I$, together with a collection of morphisms $\varphi_{ji} : c_i \to c_j$ for all $i, j \in I$ with $i \leq j$, such that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ whenever $i \leq j \leq k$.

(b) A *cocone* to a direct system $\{c_i, \varphi_{ji}\}_I$ in C is an object c of C together with a collection of morphisms $\psi_i : c_i \to c$ for each $i \in I$, such that whenever $i \leq j$, the diagram

$$\begin{tikzcd} c_i \arrow[dr, bend left, "\psi_i"] \arrow[dd, "\varphi_{ji}"'] & \\ & c \\ c_j \arrow[ur, bend right, "\psi_j"'] & \end{tikzcd} \qquad \text{commutes.}$$

(c) A cocone $\{c, \psi_i\}_I$ to a direct system $\{c_i, \varphi_{ji}\}_I$ is called a *direct limit* or *colimit*, if it satisfies the following universal property: For every other cocone c', there exists a unique morphism $c \to c'$ in C such that whenever $i \leq j$, the diagram

$$\begin{tikzcd}
c_i \arrow[dd, "\varphi_{ji}"'] \arrow[dr, "\psi_i"] \arrow[drr, bend left, "\psi'_i"] & & \\
& c \arrow[r, dashed, "\exists!"] & c' \\
c_j \arrow[ur, "\psi_j"'] \arrow[urr, bend right, "\psi'_j"'] & &
\end{tikzcd} \tag{18}$$

commutes. The direct limit is denoted by $\varinjlim c_i$ or $\operatorname{colim} c_i$.

Remark 2.28. A directed set I gives rise to a category with objects the elements of I and precisely one morphism $i \to j$ if $i \leq j$. From this point of view, a direct system in C is just a functor $I \to C$ (i.e. a *diagram in* C), and the cocone and colimit coincide with the corresponding notions in category theory.
Any directed set is in particular a *filtered category*. The latter is slightly more general in that one drops the assumption that there is at most one morphism between any two objects; instead one requires the existence of *equalizers* for each pair of parallel morphisms $\alpha, \alpha' : i \to j$, i.e. a morphism $\beta : j \to k$ such that $\beta \circ \alpha = \beta \circ \alpha'$. All results below are true with general filtered diagrams instead of directed sets, but it is usual in this context (and slightly more convenient) to restrict to directed sets.

Example 2.29 (Direct limits of sets)**.** In the category of sets, direct limits always exist. If $\{c_i, \varphi_{ji}\}_I$ is a direct system of sets, a direct limit c can be constructed explicitly by

$$c = \coprod_{i \in I} c_i \Big/ \sim \tag{19}$$

where for $x \in c_i$, $y \in c_j$, we declare $x \sim y$ if and only if there exist $k \in I$ such that $\varphi_{ki}(x) = \varphi_{kj}(x)$. The maps $\psi_i : c_i \to c$ are just the obvious maps $\psi_i(x) = [x]$.

Example 2.30 (Filtered colimits of algebraic structures)**.** If C is a category of algebraic structures such as the category of semigroups, groups, algebras or $*$-algebras, direct limits always exist. To realize the direct limit of a direct system $\{c_i, \varphi_{ji}\}_I$, use the construction in Example 2.29 of the colimit c as a set and observe that one obtains induced algebraic structures on it in a (semi-)obvious way. For example, if each of the objects c_i is a semigroup, we obtain a well-defined multiplication on the set c defined in (19) by defining $[x] \cdot [y] = [\varphi_{ki}(x) \cdot \varphi_{kj}(y)]$ for $x \in c_i$, $y \in c_j$, where $k \in I$ is such that $i \leq k$ and $j \leq k$.

Example 2.31 (Filtered colimits of C^*-algebras)**.** In the category of C^*-algebras, direct limits exist. To realize the direct limit of a direct system $\{A_i, \Phi_{ji}\}_I$, start with the colimit $\mathcal{A}$ in the category of $*$-algebras (see Example 2.30), and define a seminorm on $\mathcal{A}$ as follows. For $[a] \in \mathcal{A}$ represented by $a \in A_i$, set

$$\|[a]\| := \inf\{\|\Phi_{ji}(a)\| \mid i \leq j\}, \tag{20}$$

where we take the infimum over all $j \in I$ with $j \geq i$. One checks that this seminorm is independent of the choice of a. Since $*$-homomorphisms are always contractive by Prop. 1.16, the seminorm is finite (for non-unital algebras, the same result is true, after passing to the unitalization). It satisfies the C^*-identity, as induced from that of the A_i. The completion of A with respect to this norm is the required direct limit (note that the canonical map $\mathcal{A} \to A$ is not necessarily injective as the seminorm may be degenerate).

Example 2.32. Let $\{A_i, \Phi_{ji}\}_I$ be a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$. Then $\{M_n(A), M_n(\Psi_i)\}_I$ is the direct limit of the direct system $\{M_n(A_i), M_n(\Phi_{ji})\}_I$ of C^*-algebras.

Lemma 2.33. Let A be a C^*-algebra and $a \in A$ be self-adjoint with $\|a^2 - a\| \leq \varepsilon$ for some $\varepsilon < \frac{1}{4}$. Then there exists a projection $p \in A$ such that $\|a - p\| \leq 2\varepsilon$.

Proof. Let $B \subset A$ be the C^*-subalgebra generated by a. Since a is self-adjoint, B is commutative and by Thm. 1.30, $B \cong C(\sigma(a))$. Let $f(\lambda) = \lambda^2 - \lambda$. Then

$$\varepsilon \geq \|a^2 - a\| = \|f(a)\| = \sup_{\lambda \in \sigma(a)} |f(\lambda)|.$$

Some analysis shows that (provided $\varepsilon \leq \frac{1}{4}$), we have $|\lambda^2 - \lambda| \leq \varepsilon$ if and only if

$$\lambda \in [-\delta', \delta] \cup [1-\delta, 1+\delta'], \qquad \text{with} \qquad \delta = \frac{1}{2}\left(1 - \sqrt{1-4\varepsilon}\right), \quad \delta' = \frac{1}{2}\left(\sqrt{1+4\varepsilon} - 1\right).$$

Notice that as functions of ε, we have $\delta, \delta' \le 2\varepsilon$ whenever $\varepsilon \le \frac{1}{4}$. We conclude that $\sigma(a) \subset [-\delta', \delta] \cup [1-\delta, 1+\delta']$, and that if $\varepsilon < \frac{1}{4}$, then $\frac{1}{2} \notin \sigma(a)$. Therefore, the function

$$H(\lambda) = \begin{cases} 0 & \text{if } \lambda \le \frac{1}{2} \\ 1 & \text{if } \lambda > \frac{1}{2} \end{cases} \qquad \text{satisfies} \qquad \sup_{\lambda \in \sigma(a)} |H(\lambda) - \lambda| \le 2\varepsilon,$$

and is continuous on $\sigma(a)$, that is $H \in C(\sigma(a))$. Therefore the self-adjoint element $p := H(a) \in A$ satisfies $\|p - a\| \le 2\varepsilon$ and because $H^2 = H$, p is a projection. □

Corollary 2.34. Let $\{A_i, \Phi_{ji}\}_I$ be a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$. For each projection $p \in A$ and any $\varepsilon > 0$, there exists $i \in I$ and a projection $p_i \in A_i$ such that $\|p - \Psi_i(p_i)\| \le \varepsilon$.

Proof. By the explicit description of A (see Example 2.31), there exists a sequence $a_n \in A_{i_n}$, $n \in \mathbb{N}$ such that $\Psi_{i_n}(a_n) \to p$ in A. Since p is self-adjoint (after possibly replacing a_n by $\frac{1}{2}(a_n + a_n^*)$) we may assume that a_n is self-adjoint. As multiplication in A is continuous, the sequence $\Psi_{i_n}(a_n)^2$ converges to p^2 in A. Therefore, given any $\varepsilon > 0$, we can choose $n \in \mathbb{N}$ large enough so that both

$$\|\Psi_{i_n}(a_n)^2 - p^2\| < \frac{\varepsilon}{5}, \qquad \text{and} \qquad \|\Psi_{i_n}(a_n) - p\| < \frac{\varepsilon}{5}.$$

Using that $p = p^2$ is a projection, we therefore get

$$\|\Psi_{i_n}(a_n^2 - a_n)\| \le \|\Psi_{i_n}(a_n)^2 - p^2\| + \|\Psi_{i_n}(a_n) - p\| < \frac{2\varepsilon}{5}$$

By the definition of the norm of A, there exists $j \ge i_n$ such that $a := \Phi_{ji_n}(a_n) \in A_j$ also satisfies $\|a^2 - a\| < \frac{2\varepsilon}{5}$. Therefore by Lemma 2.33, there exists a projection $p_j \in A_j$ such that $\|a - p_j\| \le \frac{4\varepsilon}{5}$. With this projection,

$$\begin{aligned} \|p - \Psi_j(p_j)\| &\le \|p - \Psi_j \Phi_{ji_n}(a_n)\| + \|\Psi_j(a) - \Psi_j(p_j)\| \\ &\le \|p - \Psi_{i_n}(a_n)\| + \|a - p_j\| \le \frac{\varepsilon}{5} + \frac{4\varepsilon}{5} = \varepsilon. \end{aligned}$$

where we used that $*$-homomorphisms between C^*-algebras are contractive (Prop. 1.16). □

Proposition 2.35. If $\{A_i, \Phi_{ji}\}_I$ is a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$, then the collection $\{\mathcal{V}(A_i), \mathcal{V}(\Phi_{ji})\}_I$ is a direct system of semigroups, and

$$\mathcal{V}(A) \cong \varinjlim \mathcal{V}(A_i).$$

Proof. We verify the universal property. To this end, let $\{S, \psi_i\}_I$ be a cocone to the direct system of semigroups $\{\mathcal{V}(A_i), \mathcal{V}(\Phi_{ji})\}_I$. We have to show that there exists a unique semigroup homomorphism $\chi : \mathcal{V}(A) \to S$ such that $\chi \circ \mathcal{V}(\Psi_i) = \psi_i$ for all $i \in I$.

On elements $x \in \mathcal{V}(A)$ of the form $x = [\Psi_i(p_i)]$ for some projection $p_i \in M_n(A_i)$, this homomorphism must be given by

$$\chi(x) = \chi([\Psi_i(p_i)]) = \chi \circ \mathcal{V}(\Psi_i)([p_i]) = \psi_i([p_i]). \tag{21}$$

But by Corollary 2.34, any element $x \in \mathcal{V}(A)$ is of this form: Indeed, if $x = [p]$ for some projection $p \in M_n(A)$, then (since $M_n(A) = \varinjlim M_n(A_i)$ by Example 2.32), Corollary 2.34 provides the existence of a projection $p_i \in M_n(A_i)$ such that $\|p - \Psi_i(p_i)\| < \frac{1}{4}$. By Lemma 2.5 and Prop. 2.6, we therefore have $p \sim \Psi_i(p_i)$, in other words $[p] = [\Psi_i(p_i)]$.

This shows that χ is uniquely determined by (21), and it is compatible with the maps $\mathcal{V}(\Psi_i)$ and ψ_i by construction. It is also easy to see that it is additive. It therefore only remains show that χ is indeed unambiguously defined by (21). In other words, we have to show that if $p_i \in M_n(A_i)$ and $p_j \in M_n(A_j)$ are two projections such that $[\Psi_i(p_i)] = [\Psi_j(p_j)]$, then $\psi_i([p_i]) = \psi_j([p_j])$.

Assume first that $\|\Psi_i(p_i) - \Psi_j(p_j)\| < \frac{1}{2}$. Then by the definition of the norm of the direct limit $M_n(A)$, there exists $k \geq i, j$ such that also $\|\Phi_{ki}(p_i) - \Phi_{kj}(p_j)\| < \frac{1}{2}$, which (again by Lemma 2.5 and Prop. 2.6) implies that $\Phi_{ki}(p_i) \sim \Phi_{kj}(p_j)$, hence

$$\psi_i([p_i]) = \psi_k \circ \mathcal{V}(\Phi_{ki}([p_i]) = \psi_k([\Phi_{ki}(p_i)]) = \psi_k([\Phi_{kj}(p_j)]) = \cdots = \psi_j([p_j]).$$

In general, (after possibly increasing matrix dimensions), let $(q_t)_{t\in[0,1]}$ a homotopy of projections with $q_0 = \Psi_i(p_i)$ and $q_1 = \Psi_j(p_j)$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ with $\|q_{t_k} - q_{t_{k-1}}\| < \frac{1}{4}$ and projections $p_{i_k} \in A_{i_k}$ with $\|\Psi_{i_k}(p_{i_k}) - q_{t_k}\| < \frac{1}{4}$; here we let $p_{i_0} = p_i$ and $p_{i_n} = p_j$. Now

$$\|\Psi_{i_k}(p_{i_k}) - \Psi_{i_{k-1}}(p_{i_{k-1}})\| \leq \|\Psi_{i_k}(p_{i_k}) - q_{t_k}\| + \|q_{t_k} - q_{t_{k-1}}\| + \|q_{t_{k-1}} - \Psi_{i_{k-1}}(p_{i_{k-1}})\| < \frac{1}{2}.$$

Hence by the previous step, $\psi_{i_k}(p_{i_k}) = \psi_{i_{k-1}}(p_{i_{k-1}})$ for all $k = 1, \ldots, n$, which finishes the proof. □

Theorem 2.36 (Continuity)**.** K-theory commutes with direct limits. In other words, if $\{A_i, \Phi_{ji}\}_I$ is a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$, then the collection $\{K_0(A_i), K_0(\Phi_{ji})\}_I$ is a direct system of groups, and

$$K_0(A) = K_0\big(\varinjlim A_i\big) \cong \varinjlim K_0(A_i).$$

Proof. By functoriality of K_0, $\{K_0(A_i), K_0(\Phi_{ji})\}_I$ and $\{G\mathcal{V}(A^+), G\mathcal{V}(\Phi_{ji}^+)\}_I$ are direct systems of abelian groups. Moreover, we have the constant direct system $\{G\mathcal{V}(\mathbb{C}), \mathrm{id}\}_I$ (of course, $G\mathcal{V}(\mathbb{C}) \cong \mathbb{Z}$, but we don't need this fact). Putting these together, we obtain a short exact sequence of direct systems of abelian groups, i.e. for each $i \leq j$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(A_i) \arrow[r] \arrow[d, "K_0(\Phi_{ji})"] & G\mathcal{V}(A_i^+) \arrow[r, "G\mathcal{V}(\varepsilon_{A_i})"] \arrow[d, "G\mathcal{V}(\Phi_{ji}^+)"] & G\mathcal{V}(\mathbb{C}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_0(A_j) \arrow[r] & G\mathcal{V}(A_j^+) \arrow[r, "G\mathcal{V}(\varepsilon_{A_j})"'] & G\mathcal{V}(\mathbb{C}) \arrow[r] & 0.
\end{tikzcd}$$

This is just the definition of K_0; the compatibility of these diagrams for three indices $i \leq j \leq k$ is just its functoriality, see the proof of Lemma 2.18. It is now well-known that such a short exact sequence of direct systems yields a short exact sequence of the direct limits, that is, we get a short exact sequence

$$0 \longrightarrow \varinjlim K_0(A_i) \longrightarrow \varinjlim G\mathcal{V}(A_i^+) \longrightarrow \varinjlim G\mathcal{V}(\mathbb{C}) \longrightarrow 0 \tag{22}$$

The term on the right is just isomorphic $G\mathcal{V}(\mathbb{C})$, as the corresponding direct system is constant. To identify the middle term, we use that both the unitalization functor and the Grothendieck functor are left adjoints, as noted in Remark 1.21 and Remark 2.10, and it is a standard fact from category theory that left adjoints commute with direct limits. Moreover, the functor $\mathcal{V}$ commutes with direct limits by Prop. 2.35. We conclude that

$$\varinjlim G\mathcal{V}(A_i^+) = G\mathcal{V}(\varinjlim A_i^+) = G\mathcal{V}(A^+).$$

Put together, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim K_0(A_i) \arrow[r] \arrow[d] & \varinjlim G\mathcal{V}(A_i^+) \arrow[r] \arrow[d, equal] & \varinjlim G\mathcal{V}(\mathbb{C}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] & G\mathcal{V}(A^+) \arrow[r] & G\mathcal{V}(\mathbb{C}) \arrow[r] & 0
\end{tikzcd} \tag{23}$$

with exact rows. By the five lemma, the canonical map $\varinjlim K_0(A_i) \to K_0(A)$ is an isomorphism. $\square$

2.6 Stability

Proposition 2.37. Let $\{A_i, \Phi_{ji}\}_I$ be a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$. Suppose that each of the structure maps $\Phi_{ji} : A_i \to A_j$ is injective. Then for any C^*-algebras B, $\{A_i \otimes B, \Phi_{ji} \otimes \mathrm{id}_B\}_I$ is a direct system of C^*-algebras with direct limit $\{A \otimes B, \Psi_i \otimes \mathrm{id}_B\}_I$.

Proof. To begin with, let $A^\circ = \bigcup_{i \in I} \Psi_i(A_i)$ be the direct limit of $\{A_i, \Phi_{ji}\}_I$ in the category of $*$-algebras. We claim that $A^\circ \otimes_{\mathrm{alg}} B$ is dense in $A \otimes B$. Indeed, if $a \in A$ is the limit of a sequence $(a_n)_{n \in \mathbb{N}}$ in A°, then for any $b \in B$, $a_n \otimes b$ converges to $a \otimes b$ in $A \otimes B$, as by (8),

$$\|a_n \otimes b - a \otimes b\| = \|(a_n - a) \otimes b\| \leq \|a_n - a\|\|b\|.$$

Hence the closure of $A^\circ \otimes_{\mathrm{alg}} B$ in $A \otimes B$ contains $A \otimes_{\mathrm{alg}} B$, which is dense by the definition of $A \otimes B$.
Clearly $\{A \otimes B, \Phi_i \otimes \mathrm{id}_B\}_I$ is a cocone. To verify the universal property, let $\{C, \Psi'_i\}_I$ be another cocone; we have to define a $*$-homomorphism $\Xi : A^\circ \to B$ such that $\Xi \circ \Psi_i = \Psi'_i$ for all $i \in I$. Clearly, on the subset $A^\circ \otimes_{\mathrm{alg}} B$, Ξ must be given by

$$\Xi(\Psi_i(a) \otimes b) = \Psi'_i(a \otimes b), \quad \text{for } a \in A_i, b \in B.$$

One easily verifies that this gives a well-defined $*$-homomorphism $\Xi : A^\circ \otimes_{\mathrm{alg}} B \to C$, which satisfies $\Xi \circ \Psi_i = \Psi'_i$ by construction. We have to verify that Ξ is continuous with respect to the spatial norm. To this end, let $x = \sum_{n=1}^m \Psi_{i_n}(a_n) \otimes b_n \in A^\circ \otimes_{\mathrm{alg}} B$, for $a_n \in A_{i_n}$ and $b_n \in B$. Then there exists $i \in I$ with $i \geq i_n$ for all $n = 1, \dots, m$, hence

$$x = \sum_{n=1}^m \Phi_i(\Phi_{ii_n}(a_n)) \otimes b_n = (\Psi_i \otimes \mathrm{id}_B)\Big(\sum_{n=1}^m \Phi_{ii_n}(a_n) \otimes b_n\Big). \tag{24}$$

The fact about the spatial tensor product we use now is that since Ψ_i is assumed to be injective, so is $\Phi_i \otimes \mathrm{id}_B$ (Corollary 1.40). Therefore,

$$\|\Xi(x)\| = \Big\|\Psi_i\Big(\sum_{n=1}^m \Phi_{ii_n}(a_n) \otimes b_n\Big)\Big\| \leq \Big\|\sum_{n=1}^m \Phi_{ii_n}(a_n) \otimes b_n\Big\| = \|x\|$$

Here we used that $*$-homomorphisms are contractive (Prop. 1.16) together with (24) and the fact that $\Phi_i \otimes \mathrm{id}_B$ is isometric (Corollary 1.32). □

Corollary 2.38. Let A be a C^*-algebra. Then the completion of the infinite matrix algebra $M_\infty(A)$ with respect to the C^*-norm induced by the inclusions $M_n(A) \to M_\infty(A)$

is isomorphic to the spatial tensor product $A \otimes \mathbb{K}$, where $\mathbb{K}$ is the algebra of compact operators on an (infinite-dimensional) separable Hilbert space.

Proof. By Lemma 1.38, we have $M_n(A) = A \otimes M_n(\mathbb{C})$, and the embeddings $M_n(A) \to M_m(A)$ for $m \geq n$ take the form $\mathrm{id}_A \otimes J_{mn}$, where $J_{mn} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is the canonical inclusion.
The limit of the direct system $\{M_n(\mathbb{C}), J_{mn}\}_{\mathbb{N}}$ in the category of $*$-algebras is $M_\infty(\mathbb{C})$, which can be identified with a dense subalgebra of $\mathbb{F}(H)$, the algebra of finite rank operators on $H = \ell^2(\mathbb{N})$, and the induced norm is just the operator norm. Hence the C^*-algebraic direct limit is its closure, the space of compact operators,

$$\varinjlim M_n(\mathbb{C}) = \mathbb{K}(H).$$

The result now follows from Prop. 2.37. □

Theorem 2.39 (Stability)**.** Let A be a C^*-algebra. Then the inclusion $\mathbb{C} \to \mathbb{K} = \mathbb{K}(H)$ as rank one operators induces an isomorphism

$$K_0(A) \cong K_0(A \otimes \mathbb{K}).$$

Proof. Consider the direct system $\{M_n(A), J_{mn}\}_{\mathbb{N}}$, with direct limit $\{A \otimes \mathbb{K}(H), J_n\}_{\mathbb{N}}$ (Corollary 2.38). Therefore, by Continuity of K_0, Thm. 2.36,

$$\varinjlim K_0(M_n(A)) = K_0(A \otimes \mathbb{K}(H)).$$

On the other hand, by construction of K_0, each of the maps $K_0(J_{mn}) : K_0(M_n(A)) \to K_0(M_m(A))$ are isomorphisms. Hence $\{M_n(A), J_{mn}\}_{\mathbb{N}}$ is the constant direct system, with each turn isomorphic to $K_0(A)$ and the connecting maps the identity under this identification. The result follows. □

3 The K-theory long exact sequence

Let A be a C^*-algebra and $J \subset A$ a closed ideal, leading to the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \longrightarrow 0. \tag{25}$$

In this section, we construct the long exact sequence of K-theory corresponding to this short exact sequence. Throughout, we denote the projection map on the quotient by $\pi : A \to A/J$ and the inclusion map of the ideal by $\iota : J \to A$.

Notice that associated to the short exact sequence (25), we also have the short exact sequences

$$
\begin{aligned}
0 \longrightarrow M_n(J) \xrightarrow{\iota} M_n(A) \xrightarrow{\pi} M_n(A/J) \longrightarrow 0 \\
0 \longrightarrow J \xrightarrow{\iota} A^+ \xrightarrow{\pi^+} (A/J)^+ \longrightarrow 0
\end{aligned}
\tag{26}
$$

There are several further short exact sequences derived from this one, see §3.2.

3.1 Half-Exactness

Definition 3.1 (Homotopy of unitaries)**.** Let A be a unital C^*-algebra. Two unitaries $u, v \in A$ are *homotopic*, denoted by $u \sim_h v$, if there exists a continuous path of unitaries $(u_t)_{t\in[0,1]}$ such that $u_0 = u$ and $u_1 = v$.

Lemma 3.2 (Lifting unitaries)**.** Let A be a unital C^*-algebra and $J \subset A$ a closed ideal. Then for any unitary $u \in A/J$ with $u \sim_h \mathbf{1}$, there exists a unitary $\widetilde{u} \in A$ with $\pi(\widetilde{u}) = u$ and $\widetilde{u} \sim_h \mathbf{1}$ in A.

Proof. First assume that $\|u - \mathbf{1}\| < 2$. Then $\sigma(u)$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda - 1| < 2\}$. In particular, $-1 \notin \sigma(u)$. On the other hand $\sigma(u)$ is contained in the unit circle, as u is unitary. Therefore the complex logarithm (defined such that $\operatorname{Log} e^{i\theta} = i\theta$ for $\theta \in (-\pi, \pi)$) is a continuous function on $\sigma(u)$. Hence we may define $z := \operatorname{Log}(u) \in A/J$. z is skew-adjoint, since

$$z^* = \operatorname{Log}(u)^* = \operatorname{Log}(u^*) = \operatorname{Log}(u^{-1}) = -\operatorname{Log}(u) = -z.$$

Let $\widetilde{z} \in A$, be a lift of z (which exists by surjectivity of π). We may arrange $\widetilde{z}$ to be skew-adjoint (by possibly replacing $\widetilde{z}$ by $(\widetilde{z} - \widetilde{z}^*)/2$). Then $\widetilde{u} := \exp(\widetilde{z})$ is the required lift of u. It is connected to $\mathbf{1}$ by the continuous path $(u_t)_{t\in[0,1]}$ of unitaries given by

$$u_t = \exp(t \operatorname{Log}(u)) \tag{27}$$

For a general unitary u with $u \sim_h \mathbf{1}$, let $(u_t)_{t\in[0,1]}$ be a continuous path of unitaries with $u_1 = u$ and $u_0 = \mathbf{1}_n$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$ such that $\|u_{t_i} - u_{t_{i-1}}\| < 2$ for each $i = 1, \dots, n$. Then $\|u^*_{t_{i-1}} u_{t_i} - \mathbf{1}_n\| < 2$, hence there exist lifts $\widetilde{w}_i \in A^+$ of $u^*_{t_{i-1}} u_{t_i}$. But then $\widetilde{w}_1 \cdots \widetilde{w}_n$ is a lift of u. Concatenating the paths (27) gives a continuous path of unitaries from u to $\mathbf{1}$. □

Remark 3.3. Conversely, the above proof shows that any unitary $u \in A$ with $\|u - \mathbf{1}\| < 2$ automatically satisfies $u \sim_h \mathbf{1}$, where the homotopy is implemented by the path (27)

Corollary 3.4. For any unitary $u \in A/J$, the unitary $\operatorname{diag}(u, u^*) \in M_2(A/J)$ has a unitary lift $\widetilde{w} \in M_2(A)$ with $\widetilde{w} \sim_h \mathbf{1}_{2n}$.

Proof. As seen in part (d) of the proof of Prop. 2.6, we have $\operatorname{diag}(u, u^*) \sim_h \mathbf{1}_{2n}$, hence the statement follows from Lemma 3.2. □

Theorem 3.5 (Half-exactness)**.** Let A be a C^*-algebra and $J \subset A$ a closed ideal. Then the sequence of groups

$$K_0(J) \xrightarrow{K_0(\iota)} K_0(A) \xrightarrow{K_0(\pi)} K_0(A/J)$$

is exact.

Proof. Clearly, if $x \in K_0(J)$, then by functoriality, $K_0(\pi) \circ K_0(\iota)(x) = K_0(\pi \circ \iota)(x) = 0$. Hence $\operatorname{im} K_0(\iota) \subseteq \ker K_0(\pi)$.
Let now $x \in \ker K_0(\pi)$ with $K_0(\pi)(x) = 0$. We have to show that $x = K_0(\iota)(y)$ for some $y \in K_0(J)$. According to Prop. 2.19(a), there exist a projection $p \in M_\infty(A^+)$ and $n \in \mathbb{N}$ such that $x = [p] - [\mathbf{1}_n]$ and $p - \mathbf{1}_n \in M_\infty(A)$. Since $K_0(\pi)(x) = 0$, by Prop. 2.19(b), we have $\operatorname{diag}(\pi^+(p), \mathbf{1}_k) \sim_u \mathbf{1}_{n+k}$ in $M_\infty((A/J)^+)$ for some $k \in \mathbb{N}$. Denote $p' := \operatorname{diag}(p, \mathbf{1}_k)$, and for some $m \in \mathbb{N}$ large enough, let $u \in M_m((A/J)^+)$ be a unitary such that

$$u\, \pi^+(p')\, u^* = u \begin{pmatrix} \pi^+(p) & \\ & \mathbf{1}_k \end{pmatrix} u^* = \mathbf{1}_{n+k} \in M_m((A/J)^+)$$

Let $w \in M_{2m}(A^+)$ be a unitary lift of $\operatorname{diag}(u, u^*)$, which exists by Corollary 3.4, and set

$$q := w \begin{pmatrix} p' & \\ & 0 \end{pmatrix} w^* \in M_{2m}(A^+).$$

Then by construction, $[q] - [\mathbf{1}_{n+k}]$ is another representative for x. On the other hand, we have

$$\pi^+(q) = \begin{pmatrix} u & \\ & u^* \end{pmatrix} \begin{pmatrix} p' & \\ & 0 \end{pmatrix} \begin{pmatrix} u^* & \\ & u \end{pmatrix} = \begin{pmatrix} u\, p' u^* & \\ & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n+k} & \\ & 0 \end{pmatrix},$$

so $q - \mathbf{1}_{n+k} \in M_{2m}(J)$ and $q \in M_{2m}(J^+)$. Therefore $y := [q] - [\mathbf{1}_{n+k}] \in K_0(J)$ is an element with $K_0(\iota)(y) = x$. □

3.2 Cone and suspension

Definition 3.6 (Cone and suspension)**.** Let A be a C^*-algebra.

(a) The *cone* of A is the C^*-algebra CA defined by

$$CA := \{f \in C([0,1], A) \mid f(0) = 0\}.$$

(b) The *suspension* of A is the C^*-subalgebra $SA \subseteq CA$ defined by

$$SA := \{f \in CA \mid f(1) = 0.\}$$

For $*$-homomorphisms $\Phi : A \to B$, we define $C\Phi : CA \to CB$ by $\Phi(f)(t) = \Phi(f(t))$; $S\Phi : SA \to SB$ is defined by the same formula. It is then clear that both C and S are functors sending C^*-algebras to C^*-algebras. It is straightforward to verify that both are *exact functors*, that is, applying them to the short exact sequence (25), one obtains short exact sequences

$$\begin{gathered} 0 \longrightarrow CJ \xrightarrow{C\iota} CA \xrightarrow{C\pi} C(A/J) \longrightarrow 0 \\ 0 \longrightarrow SJ \xrightarrow{S\iota} SA \xrightarrow{S\pi} S(A/J) \longrightarrow 0. \end{gathered} \tag{28}$$

Definition 3.7 (Mapping cone and cylinder)**.** Let A, B be C^*-algebras and let $\Phi : A \to B$ be a $*$-homomorphism.

(a) The *mapping cone* C_Φ of Φ is defined as

$$C_\Phi = \{(a, f) \in A \oplus CB \mid f(1) = \Phi(a)\}.$$

(b) The *mapping cylinder* Z_Φ of Φ is defined as

$$Z_\Phi = \{(a, f) \in A \oplus C([0,1], B) \mid f(0) = \Phi(a)\}.$$

The mapping cone and mapping cylinders extend to functors from the category whose objects are $*$-homomorphisms $\Phi : A \to B$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ {\scriptstyle\Psi_A}\downarrow & & \downarrow{\scriptstyle\Psi_B} \\ A' & \xrightarrow[\Phi']{} & B' \end{array} \tag{29}$$

to the category of C^*-algebras. Namely, given such a commutative diagram, one obtains a $*$-homomorphisms $\Psi : Z_\Phi \to Z_{\Phi'}$ and $\Psi : C_\Phi \to C_{\Phi'}$ by setting $\Psi(a, f) = (\Phi_A(a), \Psi_B \circ f) \in A' \oplus C([0, 1], B')$. It is easy to check functoriality with respect to concatenation of diagrams (29).

Lemma 3.8. Let A be a C^*-algebra.

(a) The mapping cone CA is contractible, that is, the inclusion $\Phi : \{0\} \to CA$ of the trivial C^*-algebra is a homotopy equivalence.

(b) For any $*$-homomorphism $\Phi : A \to B$, projection onto the first component $p_1 : Z_\Phi \to A$ is a homotopy equivalence.

Proof. (a) Let $\Phi' : CA \to \{0\}$ be the trivial map (a $*$-homomorphism) and for $t \in [0, 1]$, consider the $*$-homomorphism $\Phi_t : CA \to CA$, $f \mapsto f_t$, where $f_t(s) = f(ts)$. Then for any $f \in CA$, $t \mapsto \Phi_t(f)$ is a continuous map (by compactness of $[0, 1]$) hence $(\Phi_t)_{t\in[0,1]}$ is a homotopy with $\Phi_1 = \mathrm{id}$ and $\Phi_0 = \Phi \circ \Phi'$. It follows now from Thm. 2.24(b) that $K_0(CA) = \{0\}$.
(b) We claim that a homotopy inverse is given by $c : A \to Z_\Phi$, $a \mapsto (a, c_a)$, where $c_a \in C([0, 1], B)$ is the function with $c_a(t) \equiv \Phi(a)$ for all $t \in [0, 1]$: First, $p_1 \circ c = \mathrm{id}_A$. On the other hand, consider the family of $*$-automorphisms Ψ_s, $s \in [0, 1]$, of Z_Φ given by $\Psi_s(a, f) := (a, f_s)$, where for $f \in C([0, 1], B)$, the function $f_s \in C([0, 1], B)$ is given by

$$f_s(t) := \begin{cases} f(t-s) & s \le t \\ f(1) & s \ge t \end{cases}$$

Then $\Psi_1 = c \circ p_1$, while $\Psi_0 = \mathrm{id}_A$. Hence also $c \circ p_1$ is homotopic to the identity. $\square$

3.3 The long exact sequence

Lemma 3.9. Let A be a C^*-algebra, $J \subseteq A$ be a closed ideal and suppose that A/J is contractible. Then $K_0(\iota) : K_0(J) \to K_0(A)$ is an isomorphism.

Proof. First, notice that $K_0(\iota)$ is surjective by half-exactness of K_0 (Thm. 3.5), because $K_0(A/J) = \{0\}$ by contractibility of A/J and homotopy invariance of K_0 (Thm. 2.24). To show injectivity of $K_0(\iota)$, we use the mapping cylinder Z_ι. Notice here that by Lemma 3.8(b), the projection map $p_1 : Z_\iota \to J$, $(a, f) \mapsto a$ is a homotopy equivalence and hence $K_0(p_1) : K_0(Z_\iota) \to K_0(J)$ is an isomorphism (Thm. 2.24).
On the other hand, Z_ι admits a surjective $*$-homomorphism $\zeta : Z_\iota \to C_\pi$ to the mapping cone of π, given by $\zeta(a, f) = (f(1), \pi \circ f)$, with $\ker(\zeta) = CJ$. Consider the following commutative diagram with exact rows and colums, where i and p are the obvious inclusion and projection maps, in view of $C_\pi \subseteq A \oplus C(A/J)$.

$$\begin{tikzcd}
 & & & 0 & & \\
 & 0 \arrow[r] & J \arrow[r, "\iota"] & A \arrow[u] \arrow[r, "\pi"] & A/J \arrow[r] & 0 \\
0 \arrow[r] & CJ \arrow[r] & Z_\iota \arrow[u, "p_1", "\simeq"'] \arrow[r, "\zeta"'] & C_\pi \arrow[u, "p"'] \arrow[r] & 0 & \\
 & & & S(A/J) \arrow[u, "i"'] & & \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

Since CJ is contractible by Lemma 3.8(a) and $S(A/J)$ is contractible by assumption, the homomorphisms $K_0(\zeta)$ and $K_0(p)$ are injective, again by half-exactness and homotopy invariance of K_0 (Thms 3.5 & 2.24). This shows that $K_0(\iota) = K_0(p)K_0(\zeta)K_0(p_1)^{-1}$ is injective as well. □

Theorem 3.10 (Long exact sequence)**.** Let A be a C^*-algebra and let $J \subset A$ be a closed ideal. Then there exists a *boundary map* $\delta : K_0(S(A/J)) \to K_0(J)$ such that we have an exact sequence of groups

$$\begin{tikzcd}
K_0(S(A/J)) \arrow[d, "\delta"'] & K_0(SA) \arrow[l] & K_0(SJ) \arrow[l] \\
K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(A/J).
\end{tikzcd} \tag{30}$$

Moreover, the map is functorial with respect to the short exact sequence, i.e. if $\Phi : A \to A'$ is a $*$-homomorphism taking J to a closed ideal $J' \subseteq A'$, then we obtain a commutative diagram

$$\begin{tikzcd}
K_0(S(A/J)) \arrow[r, "\delta"] \arrow[d] & K_0(J) \arrow[d] \\
K_0(S(A'/J')) \arrow[r, "\delta'"'] & K_0(J').
\end{tikzcd}$$

Proof. We will use the short exact sequences

$$\begin{aligned} 0 \longrightarrow S(A/J) \xrightarrow{i} C_\pi \xrightarrow{p} A \longrightarrow 0 \\ 0 \longrightarrow J \xrightarrow{j} C_\pi \longrightarrow C(A/J) \longrightarrow 0 \end{aligned} \tag{31}$$

involving the mapping cone C_π of $\pi : A \to A/J$. Again, all maps involved are just the obvious inclusion and projection maps coming from viewing $C_\pi \subset A \oplus C(A/J)$.

Definition of the boundary map: This is done with the diagram

$$\begin{tikzcd} K_0(SA) \arrow[r, "K_0(S\pi)"] & K_0(S(A/J)) \arrow[rr, dashed, "-\delta"] \arrow[dr, "K_0(i)"'] & & K_0(J) \arrow[r, "K_0(\iota)"] \arrow[dl, "K_0(j)", "\cong"'] & K_0(A). \\ & & K_0(C_\pi) & & \end{tikzcd} \tag{32}$$

Since $C(A/J)$ is contractible by Lemma 3.8(a), Lemma 3.9 implies that the map $K_0(j)$ is an isomorphism. Therefore, we can define $\delta = -K_0(j)^{-1}K_0(i)$. Naturality of δ follows from the functoriality of the cone construction. It is left to verify exactness of the top row of this diagram.

Exactness at $K_0(J)$*:* We have the commutative diagram

$$\begin{tikzcd} K_0(S(A/J)) \arrow[r, "K_0(i)"] \arrow[d, equal] & K_0(C_\pi) \arrow[r, "K_0(p)"] & K_0(A) \arrow[d, equal] \\ K_0(S(A/J)) \arrow[r, "-\delta"'] & K_0(J) \arrow[r, "K_0(\iota)"'] \arrow[u, "K_0(j)"', "\cong"] & K_0(A). \end{tikzcd}$$

The top row is exact (in the middle) by half-exactness of K_0, Thm. 3.5, applied to the first short exact sequence in (31). Since $K_0(j)$ is an isomorphism, this implies that also the bottom row is exact.

Exactness at $K_0(S(A/J))$*:* At this point we know that for any C^*-algebra A_1 with a closed ideal $J_1 \subseteq A_1$, the sequence

$$K_0(S(A_1/J_1) \xrightarrow{\delta_1} K_0(J_1) \xrightarrow{K_0(\iota_1)} K_0(A_1) \xrightarrow{K_0(\pi_1)} K_0(A_1/J_1) \tag{33}$$

is exact. The trick is to apply this to the first exact sequence in (31), that is, we set $J_1 := S(A/J)$ and $A_1 := C_\pi$, with maps $\iota_1 = i$, $\pi_1 = p$, which then gives $A_1/J_1 \cong A$. Substituting these definitions in (33), we obtain that the top row of the diagram

$$\begin{tikzcd} K_0(SA) \arrow[r, "\delta_1"] \arrow[d, equal] & K_0(S(A/J)) \arrow[r, "K_0(i)"] \arrow[d, equal] & K_0(C_\pi) \arrow[r, "K_0(p)"] & K_0(A) \arrow[d, equal] \\ K_0(SA) \arrow[r, "K_0(S\pi)"'] & K_0(S(A/J)) \arrow[r, "-\delta"'] & K_0(J) \arrow[r, "K_0(\iota)"'] \arrow[u, "K_0(j)"', "\cong"] & K_0(A) \end{tikzcd} \tag{34}$$

is exact. We have seen that the two squares on the right hand side commute; since both $K_0(j)$ and $K_0(\sigma)$ are isomorphisms, the exactness of the top row implies that of the bottom row, provided that we can verify that the left-most square commutes as well, that is, $\delta_1 = K_0(S\pi)$.
To see this, we need to look more closely at the derived short exact sequences (31) for our new exact sequence (33). These are

$$\begin{aligned} &0 \longrightarrow SA \xrightarrow{i_1} C_{\pi_1} \xrightarrow{p_1} C_\pi \longrightarrow 0 \\ \text{and} \qquad &0 \longrightarrow S(A/J) \xrightarrow{j_1} C_{\pi_1} \longrightarrow CA \longrightarrow 0, \end{aligned} \tag{35}$$

involving the mapping cone C_{π_1} of $\pi_1 : C_\pi \to A$. Since π_1 is just the projection onto the first factor of $C_\pi \subseteq A \oplus C(A/J)$, upon going through the definitions, one finds that the mapping cone can be identified with

$$C_{\pi_1} = \{(g, f) \in C(A/J) \oplus CA \mid g(1) = \pi(f(1))\},$$

in such a way that the maps i_1 and j_1 in (35) are just the obvious inclusion maps under this identification. Since $\delta_1 = -K_0(j_1)^{-1}K_0(i_1)$, we have

$$\delta_1 = K_0(S\pi) \qquad \Longleftrightarrow \qquad -K_0(i_1) = K_0(j_1 \circ S\pi).$$

The idea is therefore to construct a homotopy between the $*$-homomorphisms i_1 and $j_1 \circ S\pi$ from SA to C_{π_1}. Going through the definition, one finds

$$i_1(f) = (0, f), \qquad (j_1 \circ S\pi)(f) = (\pi \circ f, 0).$$

Consider now the collection $(\Phi_s)_{s\in[0,1]}$ of $*$-homomorphisms $\Phi_s : SA \to C_{\pi_1}$ given by

$$\Phi_s(f) = (f_s^{C(A/J)}, f_s^{CA}), \qquad \text{with} \qquad \begin{aligned} f_s^{C(A/J)}(t) &= \begin{cases} 0 & t \in [0, 1-s] \\ \pi(f(2-t-s)) & t \in [1-s, 1] \end{cases} \\ f_s^{CA}(t) &= \begin{cases} 0 & t \in [0, s] \\ f(t-s) & t \in [s, 1]. \end{cases} \end{aligned}$$

It is a homotopy with $\Phi_0 = i_1$ and $\Phi_1 = j_1 \circ S\pi \circ \sigma$, where $\sigma : SA \to SA$ is the $*$-automorphism defined by $\sigma(f)(t) = f(1-t)$. By homotopy invariance (Thm. 2.24), we therefore have $K_0(i_1) = K_0(j_1 \circ S\pi) \circ K_0(\sigma)$, or equivalently $\delta_1 = -K_0(S\pi) \circ K_0(\sigma)$. To finish the proof, one could now show that $K_0(\sigma) = -\mathrm{id}_{K_0(SA)}$, which is not too hard. However this is not necessary: Instead, one can observe that so far, we have shown that the diagram (34) commutes if one replaces the left-most identity arrow by the automorphism $-K_0(\sigma)$. But also for this new diagram, exactness of the top row implies that of the bottom row. □

Corollary 3.11 (Split-exactness)**.** Let A be a C^*-algebra with a closed ideal $J \subseteq A$ such that the short exact sequence (25) *splits*, that is, there exists a $*$-homomorphism $s : A/J \to A$ such that $\pi \circ s = \mathrm{id}_{A/J}$. Then

$$K_0(A) \cong K_0(J) \oplus K_0(A/J).$$

Proof. The splitting map s provides a group homomorphism $K_0(s) : K_0(J) \to K_0(A)$ such that $K_0(\pi)K_0(s) = \mathrm{id}_{K_0(A/J)}$. This shows that $K_0(\pi)$ must be surjective. Taking suspensions, we obtain that also $K_0(S\pi)$ is surjective, hence by exactness, $\delta = 0$. Therefore $K_0(\iota)$ is injective. Therefore, we obtain a split exact short exact sequence

$$0 \longrightarrow K_0(J) \xrightarrow{K_0(\iota)} K_0(A) \underset{K_0(s)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(A/J) \longrightarrow 0.$$

As we are in the category of abelian groups, this implies that $K_0(\iota) \oplus K_0(s) : K_0(J) \oplus K_0(A/J) \to K_0(A)$ is an isomorphism. □

Remark 3.12. Of course, if A is isomorphic to $J \oplus A/J$ as a C^*-algebra, such that ι and π are just the inclusion respectively the projection map under this identification, then the result follows directly from the definition of K_0. However, the existence of a splitting $s : A/J \to A$ does *not* imply $A \cong J \oplus A/J$ as C^*-algebras. For example, if A is non-unital, then usually A^+ is not isomorphic to the direct sum $A \oplus \mathbb{C}$. However, Corollary 3.11 implies that we always have

$$K_0(A^+) \cong K_0(A) \oplus \mathbb{Z}. \tag{36}$$

To obtain a formula for the boundary map, we need the following lemma.

Lemma 3.13. Let A be a C^*-algebra and let $f \in M_m(CA^+)$ be a projection with $f(0) = \mathbf{1}_n$, where $n \leq m$. Then there exists a unitary $u \in M_m((CA)^+)$ with $u(0) = \mathbf{1}_m$ such that $f(t) = u(t)\mathbf{1}_n u(t)^*$.

Proof. We have $f \sim_h \mathbf{1}_n$, as the path f_s defined by $f_s(t) = f(st)$ is a homotopy. By Prop. 2.6(a), this implies $f \sim_u \mathbf{1}_n$. This implies the existence of a unitary $u' \in M_m((CA)^+)$ such that $f(t) = u'(t)\mathbf{1}_n u'(t)^*$ for $t \in [0, 1]$. For $t = 0$, this implies $\mathbf{1}_n = u'(0)\mathbf{1}_n u(0)$, hence $u'(0) = \mathrm{diag}(v, w)$ for unitaries $v \in M_n(A)$, $w \in M_{m-n}(A)$ (see Lemma 5.7 below). Therefore $u(t) = u'(t)u'(0)^*$ also satisfies $f(t) = u(t)\mathbf{1}_n u(t)^*$, and $u(0) = \mathbf{1}_m$. □

Proposition 3.14. Let A be a C^*-algebra and let $J \subset A$ be a closed ideal. Then the boundary map in the long exact sequence (30) has the following explicit formula: Represent $x \in K_0(S(A/J))$ as $x = [f] - [\mathbf{1}_n]$, where $f \in M_m(S(A/J)^+)$, $m \geq n$ such that $f - \mathbf{1}_n \in M_m(S(A/J))$, and choose a projection $\widetilde{f} \in M_m((CA)^+)$, $\widetilde{f}(0) = \mathbf{1}_n$ such that $\pi^+(\widetilde{f}(t)) = f(t)$ for all $t \in [0,1]$. Then

$$\delta(x) = [\widetilde{f}(1)] - [\widetilde{f}(0)]. \tag{37}$$

Proof. To see the existence of a lift, consider f as a projection in the larger space $M_\infty(C(A/J)^+)$. Then by Lemma 3.13, there exists a unitary $u \in M_m(C(A/J)^+)$ with $u(0) = \mathbf{1}_m$ such that $f = u\mathbf{1}_n u^*$. We have $u \sim_h \mathbf{1}_m$ since the family of paths $(u_s)_{s\in[0,1]}$ defined by $u_s(t) = u(st)$ provides a homotopy; hence by Lemma 3.2, there exists a unitary lift $\widetilde{u} \in M_m(CA^+)$ of u, which automatically satisfies $\widetilde{u}(0) = \mathbf{1}_m$. Then $\widetilde{f}$ defined by $\widetilde{f}(t) := \widetilde{u}(t)\mathbf{1}_n\widetilde{u}(t)^*$ is the desired lift of f.
We have $\widetilde{f}(1) \in M_m(J^+)$, as

$$\pi^+(\widetilde{f}(1)) = \pi^+(\widetilde{u}(1))\mathbf{1}_n\pi^+(\widetilde{u}(1))^* = u(1)\mathbf{1}_n u(1)^* = f(1) = \mathbf{1}_n.$$

In particular, since $\widetilde{f}(0) = \mathbf{1}_n$, this implies $\widetilde{f}(1) - \widetilde{f}(0) \in M_n(J)$, hence $[\widetilde{f}(1)] - [\widetilde{f}(0)]$ is a well-defined element of $K_0(J)$.
Since δ is defined by $\delta = -K_0(j)^{-1} \circ K_0(i)$, in order to see the formula (37), we compare $K_0(j)([\widetilde{f}(1)] - [\widetilde{f}(0)])$ with $K_0(i)(x)$ in $K_0(C_\pi)$. Remember that the maps $j : J \to C_\pi$ and $i : S(A/J) \to C_\pi$ are given by $j(a) = (a, 0)$, respectively $i(f) = (0, f)$. Therefore,

$$j^+(\widetilde{f}(1)) = j(\widetilde{f} - \mathbf{1}_n) + j^+(\mathbf{1}_n) = (\widetilde{f}(1) - \mathbf{1}_n, 0) + (\mathbf{1}_n, \mathbf{1}_n) = (\widetilde{f}(1), \mathbf{1}_n),$$

where we use that $(\mathbf{1}, \mathbf{1})$ is the unit of $(CA)^+$. We therefore have

$$K_0(j)([\widetilde{f}(1)] - [\widetilde{f}(0)]) = [(\widetilde{f}(1), \mathbf{1}_n)] - [(\widetilde{f}(0), \mathbf{1}_n)].$$

On the other hand,

$$K_0(i)(x) = [i^+(f)] - [i^+(\mathbf{1}_n)] = [(\mathbf{1}_n, f)] - [(\mathbf{1}_n, \mathbf{1}_n)]$$

We have to show that these elements are add to zero in $K_0(C_\pi)$. This follows from the calculation

$$\begin{aligned}
[(\widetilde{f}(1), \mathbf{1}_n)] + [(\mathbf{1}_n, f)] &= \left[\left(\begin{pmatrix} \widetilde{f}(1) & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & f \end{pmatrix}\right)\right] \\
&\overset{(*)}{=} \left[\left(\begin{pmatrix} \widetilde{f}(1) & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \begin{pmatrix} f & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}\right)\right] \\
&\overset{(\dagger)}{=} \left[\left(\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}\right)\right] \\
&= 2[(\mathbf{1}_n, \mathbf{1}_n)],
\end{aligned}$$

where we have to justify the equalities $(*)$ and $(\dagger)$. For $(*)$, we use the homotopy $(q_s)_{s\in[0,1]}$ of projections in $M_{2m}(C_\pi^+)$, defined by

$$q_s = \left(\begin{pmatrix} \widetilde{f}(1) & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, r_s \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & f \end{pmatrix} r_s^* \right), \qquad \text{where} \qquad r_s = \begin{pmatrix} \cos\left(\frac{\pi s}{2}\right) & -\sin\left(\frac{\pi s}{2}\right) \\ \sin\left(\frac{\pi s}{2}\right) & \cos\left(\frac{\pi s}{2}\right) \end{pmatrix};$$

observe here that due to the fact that $f(1) = \mathbf{1}_n$, each q_s indeed defines a matrix with values in the mapping cone C_π.
For the equality $(\dagger)$, consider the homotopy $(p_s)_{s\in[0,1]}$ of projections in $M_m(C_\pi^+)$ given by

$$p_s = (\widetilde{f}_s(1), \pi \circ \widetilde{f}_s), \qquad \text{with} \qquad \widetilde{f}_s(t) = \widetilde{f}(st).$$

Then $p_0 = (\mathbf{1}_n, \mathbf{1}_n)$ and $p_1 = (\widetilde{f}(1), f)$. Stabilising this, we obtain a homotopy implementing $(\dagger)$. $\square$

4 Bott Periodicity

In this section, we prove the main result of operator K-theory, *Bott periodicity*. Throughout this section, we identify $S^n\mathbb{C} \otimes A \cong S^nA$, in view of Example 1.41.

4.1 The exterior product

Let A, B be two C^*-algebras. If $p \in M_n(A)$, $q \in M_m(B)$ are projections, then their tensor product $p \otimes q \in M_n(A) \otimes M_m(B) \cong M_{nm}(A \otimes B)$ is again a projection. Applying the $\mathcal{V}$ functor, we obtain a well-defined map

$$\times : \mathcal{V}(A) \times \mathcal{V}(B) \longrightarrow \mathcal{V}(A \otimes B), \qquad ([p],[q]) \longmapsto [p \otimes q], \tag{38}$$

which is $\mathbb{N}$-bilinear. Applying the Grothendieck construction, we obtain a $\mathbb{Z}$-bilinear product on the associated Grothendieck groups. On the category of unital C^*-algebras, where we can identify $K_0 = G\mathcal{V}$ (see Remark 2.17), this gives a product map $\times : K_0(A) \times K_0(B) \to K_0(A \otimes B)$. By construction, the map is natural in the sense that for any pair of unital $*$-homomorphisms $\Phi : A \to A'$, $\Psi : B \to B'$, the diagram

$$\begin{tikzcd} K_0(A) \times K_0(B) \arrow[r, "\times"] \arrow[d, "K_0(\Phi)\times K_0(\Psi)"'] & K_0(A \otimes B) \arrow[d, "K_0(\Phi\otimes\Psi)"] \\ K_0(A') \times K_0(B') \arrow[r, "\times"'] & K_0(A' \otimes B') \end{tikzcd} \tag{39}$$

commutes.
To extend this construction to the non-unital case, we need the following lemma.

Lemma 4.1. Let A, B be C^*-algebras. Then we naturally have

$$K_0(A^+ \otimes B^+) \cong K_0(A \otimes B) \oplus K_0(A) \oplus K_0(B) \oplus \mathbb{Z}.$$

Moreover, under this identification, we have

$$K_0(A \otimes B) = \ker(K_0(\varepsilon_A \otimes \mathrm{id}_{B^+})) \cap \ker(K_0(\mathrm{id}_{A^+} \otimes \varepsilon_B)) \subset K_0(A^+ \otimes B^+). \tag{40}$$

Proof. This follows easily from split-exactness, Corollary 3.11, as we have the split exact sequences

$$0 \longrightarrow A \otimes B^+ \longrightarrow A^+ \otimes B^+ \xrightarrow{\varepsilon_A \otimes \mathrm{id}_{B^+}} B^+ \longrightarrow 0,$$

$$0 \longrightarrow A \otimes B \longrightarrow A \otimes B^+ \xrightarrow{\mathrm{id}_A \otimes \varepsilon_B} A \longrightarrow 0,$$

as well as $K_0(A^+) = K_0(A) \oplus \mathbb{Z}$, $K_0(B^+) = K_0(B) \oplus \mathbb{Z}$, see (36). $\square$

Corollary 4.2. The product map defined above sends $K_0(A) \times K_0(B) \subset K_0(A^+) \times K_0(B^+)$ to $K_0(A \otimes B) \subset K_0(A^+ \otimes B^+)$.

Proof. If $x \in K_0(A) \subset K_0(A^+)$ and $y \subset K_0(B) \subset K_0(B^+)$, then by the naturality property (39),

$$K_0(\varepsilon_A \otimes \mathrm{id}_{B^+})(x \times y) = K_0(\varepsilon_A)(x) \times y = 0,$$
$$K_0(\mathrm{id}_{A^+} \otimes \varepsilon_B)(x \times y) = x \times K_0(\varepsilon_B)(y) = 0$$

From (40), it then follows that $x \times y \in K_0(A \otimes B)$. $\square$

We can now make the following definition.

Definition 4.3 (Exterior product)**.** Let A and B be C^*-algebras. The product

$$\times : K_0(A) \times K_0(B) \longrightarrow K_0(A \otimes B), \qquad (x, y) \mapsto x \times y, \tag{41}$$

defined above is called the *exterior product*.

Lemma 4.4 (Properties of the exterior product)**.** Let A, B be C^*-algebras.

(a) The exterior product is *natural*, in the sense that for any pair of $*$-homomorphisms $\Phi : A \to A'$, $\Psi : B \to B'$, the diagram (39) commutes.

(b) The class $1 \in \mathbb{Z} \cong K_0(\mathbb{C})$ is a two-sided *unit* for the exterior product, meaning that under the canonical isomorphisms $K_0(\mathbb{C} \otimes A) \cong K_0(A)$ and $K_0(A \otimes \mathbb{C}) \cong K_0(A)$, the elements $1 \times x$, respectively $x \times 1$ are identified with x, for any $x \in K_0(A)$.

(c) The exterior product is *commutative*, in the sense that

$$x \times y = K_0(\sigma)(y \times x), \tag{42}$$

for all $x \in K_0(A)$ and $y \in K_0(B)$, where $\sigma : A \otimes B \to B \otimes A$ is the symmetry isomorphism of the tensor product.

Proof. All of these properties are induced by the analogous properties of the product (38), for which they are easily verified. □

4.2 The Töplitz exact sequence and the Bott element

Throughout this section, we denote $\mathbb{K} = \mathbb{K}(\ell^2(\mathbb{N}))$, $\mathbb{B} = \mathbb{B}(\ell^2(\mathbb{N}))$.

Definition 4.5 (Toeplitz algebra)**.** The *Toeplitz algebra* $\mathcal{T} \subset \mathbb{B}$ is the subalgebra generated by the *shift operator*, explicitly

$$(S\alpha)_n = \begin{cases} \alpha_{n-1} & n \geq 2 \\ 0 & n = 1, \end{cases} \qquad \alpha \in \ell^2(\mathbb{N}).$$

Lemma 4.6. We have $\mathbb{K} \subset \mathcal{T}$.

Proof. Let $e_1, e_2, \dots$ be the canonical basis of $\ell^2(\mathbb{N})$. Then $\mathrm{id} - SS^* = e_1 \otimes e_1^*$, the projection onto the one-dimensional subspace spanned by e_1. More generally, we have $e_m \otimes e_n^* = S^m(\mathrm{id} - SS^*)(S^*)^n$ for any $m, n \in \mathbb{N}$. Taking the linear span of these operators, we see that $\mathcal{T}$ contains all finite rank operators. But since by definition, $\mathcal{T}$ is norm-closed, it must contain the closure of the finite rank operators, which is $\mathbb{K}$. □

Proposition 4.7. We have $\sigma(S) = \overline{\mathbb{D}} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ and $\sigma_{\mathrm{ess}}(S) = \mathbb{T}$.

Remember here that the essential spectrum is the set of number $\lambda \in \mathbb{C}$ such that $\lambda - S$ is not a Fredholm operator (see Example 1.13). We use the following criterion.

Lemma 4.8. Let H be a Hilbert space and $T \in \mathbb{B}(H)$. Given $\lambda \in \mathbb{C}$, assume that there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in H without accumulation point such that $\|v_n\| = 1$ for each $n \in \mathbb{N}$ and $\|Tv_n - \lambda v_n\| \to 0$. Then $\lambda \in \sigma_{\mathrm{ess}}(T)$.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence with $\|v_n\| = 1$ for each $n \in \mathbb{N}$ and $\|Tv_n - \lambda v_n\| \to 0$. Suppose that $\lambda - T$ is a Fredholm operator. Then there exists $S \in \mathbb{B}(H)$ such that $S(\lambda - T) = \mathrm{id}_H + K$, with $K \in \mathbb{K}(H)$. Since K is compact and $\|v_n\| = 1$ for each $n \in \mathbb{N}$, the sequence $(Kv_n)_{n\in\mathbb{N}}$ has an accumulation point $w \in H$. On the other hand, we have

$$v_n = S(\lambda - T)v_n - Kv_n.$$

Since $S(\lambda - T)v_n$ converges to zero, after passing the a subsequence, the right hand side converges to w. Therefore $(v_n)_{n\in\mathbb{N}}$ has an accumulation point. $\square$

Proof of Prop. 4.7. First of all, observe that since $\|S\| = 1$, we have $\sigma(S) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.
For each $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the sequence α with $\alpha_n = \lambda^n$ is contained in $\ell^2(\mathbb{N})$ and satisfies $S^*\alpha = \lambda\alpha$. Hence $\lambda \in \sigma(S^*)$ and $\overline{\lambda} \in \sigma(S)$. On the other hand, since $S^*S = \mathrm{id}$, the operator $T := -\sum_{n=0}^{\infty} \lambda^n (S^*)^{n+1}$ satisfies $(\lambda - S)T = \mathrm{id}$ and

$$T(\lambda - S) = -\sum_{n=0}^{\infty} \lambda^{n+1}(S^*)^{n+1} + \sum_{n=0}^{\infty} \lambda^n SS^*(S^*)^n = \mathrm{id} + (SS^* - \mathrm{id}) \sum_{n=0}^{\infty} \lambda^n (S^*)^n.$$

Since $\mathrm{id} - SS^* \in \mathbb{K}$, we see that T is a parametrix for S, so that S is Fredholm. We conclude that $\lambda \notin \sigma_{\mathrm{ess}}(\mathbb{T})$.
Let now $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. For $m \in \mathbb{N}$, define a sequence $\alpha^{(m)} \in \ell^2(\mathbb{N})$ by $\alpha_n^{(m)} = \lambda^n/\sqrt{m}$ if $n \leq m$ and $\alpha_n^{(m)} = 0$ for $n > m$. Then $\|\alpha^{(m)}\| = 1$ and

$$(\lambda - S)\alpha^{(m)} = \begin{cases} 0 & \text{if } n < m \text{ or } n > m \\ \lambda^{m+1}/\sqrt{m} & \text{if } n = m. \end{cases}$$

We obtain that $\|(\lambda - S)\alpha^{(m)}\|^2 = 1/m$, which converges to zero as $m \to \infty$. Moreover, since the sequence $\alpha^{(m)}$ converges pointwise to zero, the only possible accumulation point is zero; but this is impossible since $\|\alpha^{(m)}\| = 1$ for all $m \in \mathbb{N}$. Hence $\alpha^{(m)}$ has no accumulation point. We conclude from Lemma 4.8 that $\lambda \in \sigma_{\mathrm{ess}}(S)$. $\square$

Proposition 4.9. There exists a unique surjective $*$-homomorphism $\pi : \mathcal{T} \to C(\mathbb{T})$ such that $\pi(S) = z$, the identity function on $\mathbb{T}$. Moreover, $\ker(\pi) = \mathbb{K}$, hence we have a short exact sequence

$$0 \longrightarrow \mathbb{K} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0. \tag{43}$$

Proof. Clearly, $\mathbb{K}$ is an ideal in $\mathcal{T}$. Consider the C^*-algebra $A := \mathcal{T}/\mathbb{K}$. Since $\mathcal{T}$ is generated by S, A is generated by $[S]$. As $S^*S = \mathrm{id}$ and $\mathrm{id} - SS^* \in \mathbb{K}$, $[S] \in \mathcal{T}/\mathbb{K}$ is unitary, so A is commutative, and by Thm. 1.30, we have an isomorphism $A \cong C(\sigma([S]))$ such that $[S] \mapsto \mathrm{id}_{\sigma([S])}$.
It remains to show that $\sigma([S]) = \mathbb{T}$. By Prop. 1.29, the spectrum of $[S]$ in A is the same as the spectrum of $[S]$ in the Calkin algebra $\mathbb{B}/\mathbb{K}$. Therefore, by Prop. 4.7 $\sigma([S]) = \sigma_{\mathrm{ess}}(S) = \mathbb{T}$ (see Example 1.13). □

Indentify $S\mathbb{C} \subset C([0,1])$ with $\{f \in C(\mathbb{T}) \mid f(1) = 0\} \subset C(\mathbb{T})$ by sending $e^{2\pi i t}$ to the function $z \in C(\mathbb{T})$. Let $\mathcal{T}_0 = \ker(q)$, where $q = \mathrm{ev}_1 \circ \pi : \mathcal{T} \to \mathbb{C}$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K} \arrow[r, "\iota"] \arrow[d, equal] & \mathcal{T}_0 \arrow[r, "\pi"] \arrow[d] & S\mathbb{C} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r, "\iota"] & \mathcal{T} \arrow[r, "\pi"] & C(\mathbb{T}) \arrow[r] & 0 \end{tikzcd} \tag{44}$$

has exact rows.

Definition 4.10 (Bott element)**.** A *Bott element* is an element $\mathfrak{b} \in K_0(S^2\mathbb{C})$ such that $\delta(\mathfrak{b}) \in K_0(\mathbb{K})$ is the class defined by a rank one projection, where δ is the boundary map to the upper row in (44).

Theorem 4.11. A Bott element exists.

Proof. We identify $S\mathbb{C}$ with $\{f \in C(\mathbb{T}) \mid f(1) = 0\} \subset C(\mathbb{T})$ by sending the function $f(s) = e^{2\pi i s}$ to the identity function z on $C(\mathbb{T})$. Define a unitary $u_{\mathrm{Bott}} \in M_2(S^2\mathbb{C}^+) \subset M_2(SC(\mathbb{T})^+)$ by

$$u_{\mathrm{Bott}}(t) = \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right)\bar{z} \\ \sin\left(\frac{\pi t}{2}\right) z & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & \sin\left(\frac{\pi t}{2}\right) \\ -\sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \tag{45}$$

It satisfies $u_{\mathrm{Bott}}(0) = \mathbf{1}_2$, $u_{\mathrm{Bott}}(1) = \mathrm{diag}(\bar{z}, z)$, therefore

$$p_{\mathrm{Bott}} = u_{\mathrm{Bott}} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} u_{\mathrm{Bott}}^* \tag{46}$$

is a projection in $M_2(S^2\mathbb{C}^+)$ with $p_{\mathrm{Bott}} - \mathbf{1}_1 \in M_2(S^2\mathbb{C})$. Hence $\mathfrak{b} := [p_{\mathrm{Bott}}] - [\mathbf{1}_1]$ defines an element of $K_0(S^2\mathbb{C})$.
To calculate $\delta(\mathfrak{b})$, we use (37). The unitary $U \in M_2(S\mathcal{T}_0^+) = M_2(S\mathcal{T})$ defined by

$$U(t) = \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) S^* \\ \sin\left(\frac{\pi t}{2}\right) S & \cos\left(\frac{\pi t}{2}\right) SS^* + (\mathbf{1} - SS^*) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & \sin\left(\frac{\pi t}{2}\right) \\ -\sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}$$

is a lift of u_{Bott} with $U(0) = \mathbf{1}_2$. Hence

$$\delta(\mathfrak{b}) = \left[U(1)\begin{pmatrix}\mathbf{1} & 0\\ 0 & 0\end{pmatrix}U(1)^*\right] - \left[\begin{pmatrix}\mathbf{1} & 0\\ 0 & 0\end{pmatrix}\right] = \left[\begin{pmatrix}\mathbf{1} & 0\\ 0 & \mathbf{1}-SS^*\end{pmatrix}\right] - \left[\begin{pmatrix}\mathbf{1} & 0\\ 0 & 0\end{pmatrix}\right] = [\mathbf{1}-SS^*]$$

Since $\mathbf{1} - SS^*$ is a rank one projection in $\mathbb{K}$, the result follows. $\square$

We finish this section with the following lemma. which is needed in the next section.

Lemma 4.12. For any C^*-algebra A, the rows of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K}\otimes A \arrow[r, "\iota\otimes\mathrm{id}_A"] \arrow[d, equal] & \mathcal{T}_0\otimes A \arrow[r, "\pi\otimes\mathrm{id}_A"] \arrow[d] & SA \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{K}\otimes A \arrow[r, "\iota\otimes\mathrm{id}_A"] & \mathcal{T}\otimes A \arrow[r, "\pi\otimes\mathrm{id}_A"] & C(\mathbb{T})\otimes A \arrow[r] & 0
\end{tikzcd}$$

are exact.

Proof. It suffices to consider the second sequence. By Corollary 1.40, the map $\iota\otimes\mathrm{id}_A$ is injective, hence we naturally have $\mathbb{K}\otimes A \subset \mathcal{T}\otimes A$. It is also straightforward to see that $\mathbb{K}\otimes A$ is an ideal in $\mathcal{T}\otimes A$. Because the relation $(\pi\otimes\mathrm{id}_A)\circ(\iota\otimes\mathrm{id}_A) = 0$ is still true (since it holds on the dense subset $\mathbb{K}\otimes_{\mathrm{alg}} A \subset \mathbb{K}\otimes A$), we obtain a $*$-homomorphism

$$\Phi : (\mathcal{T}\otimes A)/(\mathbb{K}\otimes A) \longrightarrow C(\mathbb{T})\otimes A.$$

It is surjective, because the dense inclusion $C(\mathbb{T})\otimes_{\mathrm{alg}} A \subset C(\mathbb{T})\otimes A$ factors through Φ:

$$C(\mathbb{T})\otimes_{\mathrm{alg}} A \cong (\mathcal{T}\otimes_{\mathrm{alg}} A)/(\mathbb{K}\otimes_{\mathrm{alg}} A) \xrightarrow{\ \Psi\ } (\mathcal{T}\otimes A)/(\mathbb{K}\otimes A) \xrightarrow{\ \Phi\ } C(\mathbb{T})\otimes A.$$

Since $\Psi : C(\mathbb{T})\otimes_{\mathrm{alg}} A \to (\mathcal{T}\otimes A)/(\mathbb{K}\otimes A)$ is an injective $*$-homomorphism with dense image (also by the diagram above), it induces a C^*-norm $\|\cdot\|_\alpha$ on $C(\mathbb{T})\otimes_{\mathrm{alg}} A$ such that the corresponding completion $C(\mathbb{T})\otimes_\alpha A \cong (\mathcal{T}\otimes A)/(\mathbb{K}\otimes A)$. As seen, it comes with surjective $*$-homomorphism $C(\mathbb{T})\otimes_\alpha A \to C(\mathbb{T})\otimes A$, therefore $\|\cdot\|_\sigma \le \|\cdot\|_\alpha$. There are several ways to see that $\|\cdot\|_\alpha \le \|\cdot\|_\sigma$. For example, it is a fact that $C(\mathbb{T})$ is *nuclear*, meaning that all C^*-norms on $C(\mathbb{T})\otimes_{\mathrm{alg}} A$ coincide. Another approach uses the *group C^*-algebra* $C^*(\mathbb{Z})$, which is the C^*-subalgebra of $\mathbb{B}(\ell^2(\mathbb{Z}))$ generated by the unilateral translation U, defined by $(U\alpha)_n = \alpha_{n+1}$ for $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. It is commutative, and since $\sigma(U) = \mathbb{T}$, the Gelfand transform provides a canonical isomorphism to $C(\mathbb{T})$ (this is just the inverse discrete Fourier transform). Now there is a contractive linear map

$$s : C(\mathbb{T}) \cong C^*(\mathbb{Z}) \longrightarrow \mathcal{T}, \qquad f \longmapsto T_f := V\check{f}V^*,$$

where $\check{f}$ is the inverse Gelfand transform of f and $V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the orthogonal projection. s is a section of π, that is $(\pi\circ s) = \mathrm{id}_{C(\mathbb{T})}$.

The map s is not a $*$-homomorphism (so the associated K-theory sequence does not split), but is a *completely positive map*, meaning that for all $n \in \mathbb{N}$, the induced map on matrices $M_n(s)$ maps positive elements to positive elements. In particular, tensoring with id_A provides a contraction $s \otimes \mathrm{id}_A : C(\mathbb{T}) \otimes A \to \mathcal{T} \otimes A$ (a general fact about completely positive maps, which can also easily be seen from the concrete form of s). We therefore obtain a contractive linear map

$$C(\mathbb{T}) \otimes A \xrightarrow{s \otimes \mathrm{id}_A} \mathcal{T} \otimes A \longrightarrow (\mathcal{T} \otimes A)/(\mathbb{K} \otimes A) \cong C(\mathbb{T}) \otimes_\alpha A.$$

Hence $\|\cdot\|_\alpha \leq \|\cdot\|_\sigma$ and $C(\mathbb{T}) \otimes_\alpha A \cong C(\mathbb{T}) \otimes A$. □

Remark 4.13. Above, we have essentially proved the following general result for general C^*-algebras A, B and a closed ideal $J \subseteq A$: Assume that there exists a *completely positive* map $s : A/J \to A$ such that $\pi \circ s = \mathrm{id}_{A/J}$ or that A/J is nuclear. Then the short sequence

$$0 \longrightarrow J \otimes B \xrightarrow{\iota \otimes \mathrm{id}_B} A \otimes B \xrightarrow{\pi \otimes \mathrm{id}_B} A/J \otimes B \longrightarrow 0$$

is exact.

4.3 The periodicity theorem

Throughout, let $\mathfrak{b} \in K_0(S^2\mathbb{C})$ be the Bott element constructed in the proof of Thm. 4.11. In fact, it will follows from Bott periodicity, Thm. 4.15 below, that the Bott element is in fact unique; this is irrelevant for the proof of Bott periodicity, but justifies to refer to "the" Bott element henceforth.

Definition 4.14 (Bott map). For any C^*-algebra A the *Bott map* of A is the map

$$\beta_A : K_0(A) \longrightarrow K_0(S^2\mathbb{C} \otimes A) = K_0(S^2 A), \qquad x \mapsto \mathfrak{b} \times x,$$

given by taking the exterior product with the Bott element.

It is clear from naturality of the exterior product, Lemma 4.4(a), that the Bott map is natural, that is, for each $*$-homomorphism $\Phi : A \to B$ between C^*-algebras A, B, we have a commutative diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{K_0(\Phi)} & K_0(B) \\ \big\downarrow{\scriptstyle \beta_A} & & \big\downarrow{\scriptstyle \beta_B} \\ K_0(S^2A) & \xrightarrow[K_0(S^2\Phi)]{} & K_0(S^2B). \end{array} \tag{47}$$

In other words, the Bott maps assemble to a natural transformation $\beta : K_0 \Rightarrow K_0 S^2$.

Theorem 4.15 (Bott periodicity)**.** For each C^*-algebra A, the Bott map β_A is an isomorphism. In other words, the functors K_0 and $K_0 S^2$ are naturally isomorphic.

The proof of Thm. 4.15 goes by constructing an inverse transformation $\alpha : K_0 S^2 \to K_0$. This is done as follows: Tensoring the upper sequence of (44) with a given C^*-algebra A, we obtain the sequence

$$0 \longrightarrow \mathbb{K} \otimes A \longrightarrow \mathcal{T}_0 \otimes A \longrightarrow SA \longrightarrow 0. \tag{48}$$

which is exact by Lemma 4.12. By Thm. 3.10, it therefore gives rise to a long exact sequence in K-theory, the relevant part of which is

$$\cdots \longrightarrow K_0(S(\mathcal{T}_0 \otimes A)) \longrightarrow K_0(S^2 A) \xrightarrow{\delta_A} K_0(\mathbb{K} \otimes A) \longrightarrow K_0(\mathcal{T}_0 \otimes A) \longrightarrow \cdots$$

with δ_A the corresponding differential. It is natural in A, as the differential depends naturally on the exact sequence.
Let $\lambda : \mathbb{C} \to \mathbb{K}$ be the inclusion as rank one operators, so that $K_0(\lambda \otimes \mathrm{id}_A) : K_0(A) \to K_0(\mathbb{K} \otimes A)$ is an isomorphism by Thm. 2.39. We now define

$$\alpha_A : K_0(S^2 A) \to K_0(A), \qquad \text{by} \qquad \alpha_A := K_0(\lambda \otimes \mathrm{id}_A)^{-1} \circ \delta_A.$$

It is then clear that the maps α_A assemble to a natural transformation of functors $\alpha : K_0 S^2 \Rightarrow K_0$. In other words, for any $*$-homomorphism $\Phi : A \to B$ between C^*-algebras A, B, the diagram

$$\begin{tikzcd} K_0(S^2 A) \arrow[r, "K_0(S^2\Phi)"] \arrow[d, "\alpha_A"'] & K_0(S^2 B) \arrow[d, "\alpha_B"] \\ K_0(A) \arrow[r, "K_0(\Phi)"'] & K_0(B). \end{tikzcd}$$

commutes.

Lemma 4.16. For any other C^*-algebra B and $x \in K_0(S^2 A)$, $y \in K_0(B)$, we have

$$\alpha_{A \otimes B}(x \otimes y) = \alpha_A(x) \times y.$$

Proof. Observe that by the definition (32) we have $\delta_A = K_0(j_A)^{-1} \circ K_0(i_A)$, where $j_A : \mathbb{K} \otimes A \to C_{\pi \otimes \mathrm{id}_A}$ and $i_A : S(S\mathbb{C} \otimes A) \to C_{\pi \otimes \mathrm{id}_A}$ are the inclusion maps into the mapping cone. Under the canonical isomorphisms $C_{\pi \otimes \mathrm{id}_A} \cong C_\pi \otimes A$ and $S(S\mathbb{C} \otimes A) \cong S^2\mathbb{C} \otimes A$, these maps take the form $j_A = j_{\mathbb{C}} \otimes \mathrm{id}_A$ and $i_A = i_{\mathbb{C}} \otimes \mathrm{id}_A$. We obtain that

$$\alpha_A = K_0(\lambda \otimes \mathrm{id}_A)^{-1} K_0(j_{\mathbb{C}} \otimes \mathrm{id}_A)^{-1} K_0(i_{\mathbb{C}} \otimes \mathrm{id}_A).$$

The statement now follows from the naturality (39) of the product. $\square$

Proof of Thm. 4.15. We show that both $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity transformation. First, the identity $\alpha_A \circ \beta_A = \mathrm{id}$ follows from the calculation

$$(\alpha_A \circ \beta_A)(x) = \alpha_A(\mathfrak{b} \times x) = \alpha_{\mathbb{C}}(\mathfrak{b}) \times x = 1 \times x = x,$$

for $x \in K_0(A)$, where we used property (2) and then property (1) of α.
Showing the identity $\beta_A \circ \alpha_A = \mathrm{id}$ is more involved. For any C^*-algebra A, denote by

$$\sigma_A : S^2\mathbb{C} \otimes A \to A \otimes S^2\mathbb{C}, \qquad f \otimes a \longmapsto a \otimes f$$

the "flip map". Observe that for these maps, we have the identity

$$(\mathrm{id}_{S^2\mathbb{C}} \otimes \sigma_A) \circ (\sigma_{S^2\mathbb{C}} \otimes \mathrm{id}_A) = \sigma_{S^2\mathbb{C} \otimes A} : S^2\mathbb{C} \otimes S^2\mathbb{C} \otimes A \longrightarrow S^2\mathbb{C} \otimes A \otimes S^2\mathbb{C}. \tag{49}$$

The important fact is now that

$$K_0(\sigma_{S^2\mathbb{C}} \otimes \mathrm{id}_A) = \mathrm{id}. \tag{50}$$

on $K_0(S^2\mathbb{C} \otimes S^2\mathbb{C} \otimes A)$. To see this, identify $S^2\mathbb{C} \cong C_0((0,1)^2) \cong C_0(\mathbb{R}^2)$ and notice that under this identification, $\sigma_{S^2\mathbb{C}}(f) = Q^* f$, where Q is the linear map given by the matrix

$$Q \;\widehat{=}\; \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

Since this is a determinant one orthogonal matrix, it can be connected to the identity matrix by a continuous path $(Q_t)_{t \in [0,1]}$ in $SO(4)$; then $\Phi_t(f) := Q_t^* f$, $t \in [0,1]$, is a continuous family of $*$-homomorphisms with $\Phi_1 = \sigma_{S^2\mathbb{C}}$, $\Phi_0 = \mathrm{id}$. The claim now follows from homotopy invariance, Thm. 2.24.
With these preparations, we calculate using that

$$\begin{aligned} x \times \mathfrak{b} &= K_0(\sigma_{S^2\mathbb{C} \otimes A})(\mathfrak{b} \times x) && \text{Lemma 4.4(c)} \\ &= K_0(\mathrm{id}_{S^2\mathbb{C}} \otimes \sigma_A) K_0(\sigma_{S^2\mathbb{C}} \otimes \mathrm{id}_A)(\mathfrak{b} \times x) && (49) \\ &= K_0(S^2\sigma_A)(\mathfrak{b} \times x). && (50) \end{aligned} \tag{51}$$

Here we used that $\mathrm{id}_{S^2\mathbb{C}} \otimes \sigma_A \cong S^2\sigma_A$ under the identification $S^2\mathbb{C} \otimes A \cong S^2 A$. Calculating further, we get for any $x \in K_0(S^2\mathbb{C} \otimes A) \cong K_0(S^2A)$ that

$$\begin{aligned}
(K_0(\sigma_A) \circ \beta_A \circ \alpha_A)(x) &= K_0(\sigma_A)(\mathfrak{b} \times \alpha_A(x)) \\
&= \alpha_A(x) \times \mathfrak{b} \\
&= \alpha_{A \otimes S^2\mathbb{C}}(x \times \mathfrak{b}) && \text{Lemma 4.16(b)} \\
&= (\alpha_{A \otimes S^2\mathbb{C}} \circ K_0(S^2\sigma_A))(\mathfrak{b} \times x) && (51) \\
&= (K_0(\sigma_A) \circ \alpha_{S^2\mathbb{C} \otimes A})(\mathfrak{b} \times x) && \text{naturality of } \alpha \\
&= K_0(\sigma_A)(x) && \alpha \text{ left inverse to } \beta
\end{aligned}$$

Because σ_A and hence $K_0(\sigma_A)$ is an isomorphism, the result follows. □

Corollary 4.17. The Bott element is unique.

Proof. By Bott periodicity, the map $\beta_{\mathbb{C}} = K_0(\lambda)^{-1} \circ \delta_{\mathbb{C}} : K_0(\mathbb{C}) \to K_0(S^2\mathbb{C})$ is an isomorphism, that is, $K_0(S^2\mathbb{C}) = K_0(\mathbb{C}) = \mathbb{Z}$. Since $K_0(\lambda)$ is an isomorphism, $\delta_{\mathbb{C}} : K_0(S^2\mathbb{C}) \to K_0(\mathbb{K})$ is an isomorphism as well. Hence there exists a unique element $\mathfrak{b}$ such that $\delta(\mathfrak{b})$ corresponds to the element $1 \in \mathbb{Z} \cong K_0(\mathbb{K})$. □

5 The K_1-functor and the six-term exact sequence

In this chapter, we finish our exposition of the K-theory of C*-algebras by introducing the K_1-functor, which gives important interpretation for the boundary maps in the K-theory six-term sequence.

5.1 Definition of K_1

Definition 5.1 (Unitary groups)**.** Let A be a C*-algebra. For any $n \in \mathbb{N}$, write

$$\mathcal{U}_n^+(A) := \{u \in M_n(A)^+ \mid u \text{ unitary and } u = \mathbf{1}_n + a, a \in M_n(A)\}.$$

Denote by $\mathcal{U}_n^+(A)_0 \subset \mathcal{U}_n^+(A)$ the normal subgroup of those unitaries homotopic to $\mathbf{1}_n$.

There exist the obvious inclusion maps $\mathcal{U}_n^+(A) \to \mathcal{U}_{n+1}^+(A)$ given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}.$$

By $\mathcal{U}_\infty^+(A)$, we denote the union of all the $\mathcal{U}_n^+(A)$, that is, the direct limit with respect to the above inclusion maps. We have $\mathcal{U}_\infty^+(A) \subset M_\infty(A)^+$, which induces a topology on $\mathcal{U}_\infty^+(A)$.

Definition 5.2 (The K_1-functor)**.** Let A be a C^*-algebra.

(a) We define
$$K_1(A) := \mathcal{U}_\infty^+(A)/\mathcal{U}_\infty^+(A)_0.$$

(b) If B is another C^*-algebra and $\Phi : A \to B$ is a $*$-homomorphism, we define
$$K_1(\Phi) : K_1(A) \to K_1(B), \quad [u] \longmapsto [\Phi^+(u)].$$

In total, K_1 is a functor from the category of C^*-algebras to the category of groups.

Remark 5.3. If A is unital, then $A^+ = A \oplus \mathbb{C}$ and the map $u \mapsto (u - \mathbf{1}_n, 1)$ provides an isomorphism from the unitary group $\mathcal{U}_n(A)$ to $\mathcal{U}_n^+(A)$. However, even in the unital case, we need the groups $\mathcal{U}_n^+(A)$ to deal with non-unital $*$-homomorphisms $\Phi : A \to B$. Namely, for $u \in M_n(A)$ unitary, $\Phi(u)$ is in general only a partial isometry, but Φ^+ maps $\mathcal{U}_\infty^+(A)$ to $\mathcal{U}_\infty^+(B)$.

Lemma 5.4. Let A be a C^*-algebra. Elements $x, y \in K_1(A)$ coincide if and only if there exists $n \in \mathbb{N}$ and a homotopy $(u_t)_{t\in[0,1]}$ of unitaries $u_t \in \mathcal{U}_n^+(A)$ with $x = [u_0]$ and $y = [u_1]$.

Proof. First observe that two unitaries $u_0, u_1 \in \mathcal{U}_\infty^+(A)$ represent the same class in $K_1(A)$ if and only if they are homotopic: If they are homotopic, they are clearly in the same connected component, that is, in the same coset of $\mathcal{U}_\infty^+(A)_0$. Conversely, elements in same connected component can be joined by a path of unitaries.
We now prove that one can restrict to homotopies that lie in some $\mathcal{U}_n^+(A)$ throughout. Clearly, homotopies in $\mathcal{U}_n^+(A)$ give rise to homotopies in $\mathcal{U}_\infty^+(A)$. Conversely, let $(u_t)_{t\in[0,1]}$ be a homotopy in $\mathcal{U}_\infty^+(A)$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|u_{t_i} - u_{t_{i-1}}\| < 2$. Then the unitaries u_{t_i} all lie in some $\mathcal{U}_m^+(A)$, for some $m \in \mathbb{N}$. But by remark Remark 3.3, since $\|u_{t_i} - u_{t_{i-1}}\| < 2$ (this also holds within $\mathcal{U}_m^+(A)$), there exists a homotopy in $\mathcal{U}_m^+$ between $u_{t_{i-1}}$ and u_{t_i}; concatenating these homotopies gives a homotopy in $\mathcal{U}_m^+(A)$ between u_0 and u_1. $\square$

Lemma 5.5. For any C^*-algebra A, $K_1(A)$ is abelian. Moreover, if $u \in \mathcal{U}_n^+(A)$ and $v \in \mathcal{U}_m^+(A)$, then $[uv] = [\mathrm{diag}(u, v)]$ in $K_1(A)$.

Proof. Let $x, y \in K_1(A)$ and write $x = [u]$, $y = [w]$ with $u, w \in \mathcal{U}_n^+(A)$. Define elements in $\mathcal{U}_{2n}^+(A)$ by

$$r_t := \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right)\mathbf{1}_n & -\sin\left(\frac{\pi t}{2}\right)\mathbf{1}_n \\ \sin\left(\frac{\pi t}{2}\right)\mathbf{1}_n & \cos\left(\frac{\pi t}{2}\right)\mathbf{1}_n \end{pmatrix}, \qquad w_t := \begin{pmatrix} u & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} r_t \begin{pmatrix} w & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} r_t^*.$$

Notice that, while r_t is not contained in $\mathcal{U}_{2n}^+(A)$ but only in $\mathcal{U}_{2n}(A^+)$, we nevertheless have $w_t \in \mathcal{U}_{2n}^+(A)$. Then $(w_t)_{t\in[0,1]}$ is a continuous path in $\mathcal{U}_{2n}^+(A)$ with $w_0 = \operatorname{diag}(uw, \mathbf{1}_n)$ and $w_1 = \operatorname{diag}(u, w)$. Define a continuous path $(w'_t)_{t\in[0,1]}$ by swapping the roles of u and w in the formula above, so that $w'_0 = \operatorname{diag}(wu, \mathbf{1}_n)$ and $w'_1 = \operatorname{diag}(w, u)$. Finally, define a path $(v_t)_{t\in[0,1]}$ by

$$v_t = r_t \begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix} r_t^*.$$

Then $(v_t)_{t\in[0,1]}$ is a continuous path in $\mathcal{U}_{2n}^+(A)$ with $v_0 = \operatorname{diag}(u, w)$ and $v_1 = \operatorname{diag}(w, u)$. Concatenating these paths appropriately gives a continuous path of unitaries in $\mathcal{U}_{2n}^+(A)$ from $\operatorname{diag}(uv, \mathbf{1}_n)$ to $\operatorname{diag}(vu, \mathbf{1}_n)$. This proves the claim. $\square$

5.2 Identification with Suspension

Theorem 5.6. For any C^*-algebra A, there exists a canonical isomorphism

$$\eta_A : K_1(A) \longrightarrow K_0(SA)$$

such that for each $*$-homomorphism $\Phi : A \to B$, the diagram

$$\begin{tikzcd} K_1(A) \arrow[r, "K_1(\Phi)"] \arrow[d, "\eta_A"'] & K_1(B) \arrow[d, "\eta_B"] \\ K_0(SA) \arrow[r, "K_0(S\Phi)"'] & K_0(SB). \end{tikzcd} \tag{52}$$

commutes. In other words, the maps η_A assemble to a natural isomorphism of functors $\eta : K_1 \Rightarrow K_0 S$.

We will need the following lemma.

Lemma 5.7. Let A be a C^*-algebra. Let $m \geq n$ and let $w \in M_{2n}(A)$ be unitary such that

$$w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_n(A)) \cong M_{2n}(A).$$

Then there exist unitaries $u, v \in M_n(A)$, such that $w = \mathrm{diag}(u, v)$.

Proof. Let
$$w = \begin{pmatrix} u & a \\ b & v \end{pmatrix}$$
with $u, a, b, v \in M_n(A)$. We have to show that $a = b = 0$. Since w is unitary,
$$\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} = ww^* = \begin{pmatrix} u & a \\ b & v \end{pmatrix} \begin{pmatrix} u^* & b^* \\ a^* & v^* \end{pmatrix} \quad \text{in particular} \quad \begin{cases} \mathbf{1}_n & = u^*u + a^*a \\ \mathbf{1}_n & = bb^* + vv^*. \end{cases}$$
But
$$\begin{pmatrix} u & a \\ b & v \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & b^* \\ a^* & v^* \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{implies} \quad \begin{cases} \mathbf{1}_n & = u^*u \\ 0 & = bb^*. \end{cases}$$
Putting together, we get $a^*a = b^*b = 0$, hence $a = b = 0$ (this follows from the C^*-property, as $\|a\|^2 = \|a^*a\| = 0$ and similarly for b). ☐

Proof of Thm. 5.6. We will start with the definition of η_A, then show injectivity and surjectivity of η_A and then verify that the square (52) commutes.

Definition of η_A*:* For A a C^*-algebra the map $\eta_A : K_1(A) \to K_0(SA)$ is defined as follows. Given $x \in K_1(A)$, write $x = [u]$ with $u \in \mathcal{U}_n^+(A)$ and let $(w_t)_{t\in[0,1]}$ be a homotopy in $\mathcal{U}_{2n}^+(A)$ with $w_1 = \mathrm{diag}(u, u^*)$ and $w_0 = \mathbf{1}_{2n}$ (such a homotopy exists by Corollary 3.4). Then set
$$\eta_A(x) := \big[f\big] - [\mathbf{1}_n] \in K_0(SA) \qquad \text{with} \qquad f(t) = w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w_t^*.$$
Notice that indeed, $f(t)$ is a projection for every $t \in [0, 1]$ and $f(0) = f(1) = \mathbf{1}_n$, so $f \in M_{2n}(SA^+)$.
We have to check that η_A is independent from the choice of representative in $\mathcal{U}_n^+(A)$ and the choice of homotopy $(w_t)_{t\in[0,1]}$, as well as the choice of $n \in \mathbb{N}$.

(1) *Independence of* $n \in \mathbb{N}$*:* Write
$$w_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} \qquad \text{so that} \qquad f(t) = \begin{pmatrix} a_t a_t^* & a_t c_t^* \\ c_t a_t^* & d_t c_t^* \end{pmatrix}. \tag{53}$$
If we set $u' := \mathrm{diag}(u, \mathbf{1}_m)$ for $m \in \mathbb{N}$, then $(w'_t)_{t\in[0,1]}$ with
$$w'_t := \begin{pmatrix} a_t & 0 & b_t & 0 \\ 0 & \mathbf{1}_m & & 0 \\ c_t & 0 & d_t & 0 \\ 0 & 0 & 0 & \mathbf{1}_m \end{pmatrix} \tag{54}$$

is a homotopy of unitaries from $\operatorname{diag}(u', (u')^*)$ to $\mathbf{1}_{2n+2m}$, and the corresponding path of projections is

$$f'(t) := w'_t \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_m & & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (w'_t)^* = \begin{pmatrix} a_t a_t^* & 0 & a_t c_t^* & 0 \\ 0 & \mathbf{1}_m & 0 & 0 \\ c_t a_t^* & 0 & d_t c_t^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sigma \begin{pmatrix} a_t a_t^* & a_t c_t^* & 0 & 0 \\ c_t a_t^* & d_t c_t^* & 0 & 0 \\ 0 & 0 & \mathbf{1}_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sigma^*$$

$$= \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_m & 0 \\ 0 & \mathbf{1}_n & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_m \end{pmatrix} \begin{pmatrix} a_t a_t^* & a_t c_t^* & 0 & 0 \\ c_t a_t^* & d_t c_t^* & 0 & 0 \\ 0 & 0 & \mathbf{1}_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_m & 0 \\ 0 & \mathbf{1}_n & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_m \end{pmatrix}^* . \tag{55}$$

With a view on (53), this shows that $f' \sim_u \operatorname{diag}(f, \mathbf{1}_m)$ for all $t \in [0,1]$, hence

$$[f'] - [\mathbf{1}_{n+m}] = \left[\begin{pmatrix} f & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} \right] - [\mathbf{1}_{n+m}] = [f] - [\mathbf{1}_n],$$

as desired.

(2) *Independence of representative and homotopy:* Let $u' \in \mathcal{U}_n^+(A)$ with $u' \sim_h u$ and let $(w'_t)_{t\in[0,1]}$ be a homotopy in $\mathcal{U}_{2n}^+(A)$ with $w'_1 = \operatorname{diag}(u', (u')^*)$ and $w_0 = \mathbf{1}_{2n}$. We will show that the path of projections $f'(t) = w'_t \operatorname{diag}(\mathbf{1}_n, 0)(w'_t)^*$ is unitary equivalent to the path f.
To this end, let $(u_t)_{t\in[0,1]}$ be a homotopy in $\mathcal{U}_n^+(A)$ with $u_0 = u$ and $u_1 = u'$ (here we need to possibly increase n before). Set now $v(t) = w_t \operatorname{diag}(u^* u_t, u u_t^*)(w'_t)^*$. Then $v(t)$ is unitary for each $t \in [0,1]$, with $v(0) = \mathbf{1}_{2n}$ and

$$v(1) = w_1 \begin{pmatrix} u^* u_1 & 0 \\ 0 & u u_1^* \end{pmatrix} (w'_1)^* = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} u^* u' & 0 \\ 0 & u(u')^* \end{pmatrix} \begin{pmatrix} (u')^* & 0 \\ 0 & u' \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Hence v is a unitary element in $M_{2n}(SA^+)$. Moreover,

$$v(t) f'(t) v(t)^* = w_t \begin{pmatrix} u^* u_t & 0 \\ 0 & u u_t^* \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_t^* u & 0 \\ 0 & u_t u^* \end{pmatrix} w_t^* = w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w_t^* = f(t).$$

This shows that $f' \sim_u f$ in $M_{2n}(SA^+)$.

Homomorphism property: If $x, x' \in K_1(A)$, represent them by unitaries $u, u' \in U_n^+(A)$. By Lemma 5.5, we have $x + x' = \operatorname{diag}(u, u')$. Let $(w_t)_{t\in[0,1]}$ and $(w'_t)_{t\in[0,1]}$ be homotopies of unitaries in $M_{2n}(A^+)$ with $w_0 = w'_0$ and $w_1 = \operatorname{diag}(u, u^*)$, $w'_1 = \operatorname{diag}(u', (u')^*)$ and let $f, f' \in M_{2n}(A^+)$ be the corresponding projections so that $\eta_A(x) = [f] - [\mathbf{1}_n]$, $\eta_A(x') = [f'] - [\mathbf{1}_n]$
We define $v_t = s \operatorname{diag}(w_t, w'_t) s^*$, where $s \in M_{4n}(\mathbb{C}) \subset M_{4n}(A^+)$ is the permutation matrix that previously appeared in (55). This gives a homotopy $(v_t)_{t\in[0,1]}$ with $v_0 = \mathbf{1}_{4n}$ and $v_1 = (\operatorname{diag}(u, u', u^*, (u')^*))$. Hence

$$\eta_A(x+y) = [g] - [\mathbf{1}_{2n}], \qquad \text{where} \qquad g(t) = v_t \begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & 0 \end{pmatrix} v_t^*.$$

But

$$v_t \begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & 0 \end{pmatrix} v_t^* = s \begin{pmatrix} w_t & 0 \\ 0 & w_t' \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & & & \\ & 0 & & \\ & & \mathbf{1}_n & \\ & & & 0 \end{pmatrix} \begin{pmatrix} w_t^* & 0 \\ 0 & (w_t')^* \end{pmatrix} s^* = s \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} s^*,$$

hence, since the loop constant equal to s defines an element of $M_{4n}(\mathbb{C}) \subset M_{4n}((SA)^+)$, we have

$$\eta_A(x + x') = [g] - [\mathbf{1}_{2n}] = [f] - [\mathbf{1}_n] + [f'] - [\mathbf{1}_n] = \eta_A(x) + \eta_A(x'),$$

as desired.

Injectivity: Let $x \in K_1(A)$ with $\eta_A(x) = 0$. Represent $x = [u]$ with $u \in \mathcal{U}_n^+(A)$ and let $f(t) = w_t \operatorname{diag}(\mathbf{1}_n, 0) w_t^* \in M_{2n}(A^+)$, where f is a homotopy from $\operatorname{diag}(u, u^*)$ to $\mathbf{1}_{2n}$ in $M_{2n}(A)$. Then

$$0 = \eta_A(x) = [f] - [\mathbf{1}_n]$$

in $K_0(SA)$.
We first treat the special case that $f \sim_u \operatorname{diag}(\mathbf{1}_n, 0)$ in $M_{2n}((SA)^+)$. This means that there exists a homotopy $(v_t)_{t\in[0,1]}$ of unitaries in $M_{2n}(A^+)$ with $v_0 = v_1 = \mathbf{1}_{2n}$ and for all $t \in [0, 1]$,

$$\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} = v_t f(t) v_t^* = v_t w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} (w_t v_t)^*.$$

By Lemma 5.7, $v_t w_t$ has the form $v_t w_t = \operatorname{diag}(u_t, u_t')$ for homotopies of unitaries $(u_t)_{t\in[0,1]}$, $(u_t')_{t\in[0,1]}$. By construction, $u_0 = \mathbf{1}_n$ and $u_1 = u$, so that $(u_t)_{t\in[0,1]}$ implements $u \sim \mathbf{1}_n$. Therefore $x = [u] = 0$.
We finish by showing that the general case can be reduced to the special case just treated. In general, $[f] - [\mathbf{1}_n] = 0$ only means that $\operatorname{diag}(f, \mathbf{1}_m) \sim_u \mathbf{1}_{n+m}$ for some $m \in \mathbb{N}$ and all $t \in [0, 1]$. Write $u' = \operatorname{diag}(u, \mathbf{1}_m) \in M_{n+m}(A^+)$ (which is also a representative for x) and let $(w_t')_{t\in[0,1]}$ be the homotopy from $\operatorname{diag}(u', (u')^*)$ to $\mathbf{1}_{2n+2m}$ given in (54). Then as calculated in (55),

$$\operatorname{diag}(f(t), \mathbf{1}_m) \sim_u w_t' \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_m & & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (w_t')^* =: f'(t).$$

Thus

$$\eta_A(x) = [f] - [\mathbf{1}_n] = \big[\big(\operatorname{diag}(f, \mathbf{1}_m)\big)\big] - [\mathbf{1}_{n+m}] = [f'] - [\mathbf{1}_{n+m}],$$

where by the choice of f', we have $f' \sim_u \mathbf{1}_{n+m}$ in $M_{2n+2m}((SA)^+)$. This reduces to the special case above.

Surjectivity: Let $y \in K_0(SA)$. By Prop. 2.19(a), we can represent $y = [f] - [\mathbf{1}_n]$ for some $n \in \mathbb{N}$ and some projection $f \in M_\infty((SA)^+)$ with $f - \mathbf{1}_n \in M_\infty(SA)$. As discussed in

Prop. 3.14 there exists $m \geq n$ and a homotopy $(w_t)_{t\in[0,1]}$ of unitaries in $M_m(A^+)$ such that $f(t) = w_t \operatorname{diag}(\mathbf{1}_n, 0)w_t^*$ for all $t \in [0,1]$. Moreover, we may assume that $m = 2n$ (otherwise represent $y = [(\operatorname{diag}(f, \mathbf{1}_k))] - [\mathbf{1}_{n+k}]$ for some suitable k instead). We now have

$$\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} = f(1) = w_1 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w_1^*,$$

so by Lemma 5.7, $w_1 = \operatorname{diag}(u, v)$ for unitaries $u, v \in \mathcal{U}_n^+(A)$. If now $(w'_t)_{t\in[0,1]}$ is a homotopy of unitaries with $w'_0 = \mathbf{1}_{2n}$ and $w'_1 = \operatorname{diag}(u, u^*)$, then by definition, we have

$$\eta_A([u]) = [f'] - [\mathbf{1}_n], \qquad \text{where} \qquad f'(t) = w'_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} (w'_t)^*.$$

To see that y is in the image of η_A, we will show that $[f'] - [\mathbf{1}_n] = y$.
Suppose first that $v \sim_h u^*$ in $\mathcal{U}_n^+(A)$. Then there exists a homotopy $(s_t)_{t\in[0,1]}$ of unitaries with $s_0 = \mathbf{1}_n$ and $s_1 = v^*u^*$. Therefore

$$\begin{aligned} f(t) = w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w_t^* &= w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & s_t \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & s_t \end{pmatrix}^* w_t^* \\ &= w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & s_t \end{pmatrix} (w'_t)^* f'(t)\, w'_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & s_t \end{pmatrix}^* w_t^* \end{aligned}$$

for all $t \in [0,1]$. Now one easily checks that homotopy $(w_t \operatorname{diag}(\mathbf{1}_n, s_t)(w'_t)^*)$ starts and ends at $\mathbf{1}_{2n}$, hence defines a unitary element in $M_{2n}((SA)^+)$ and implements $f \sim_u f'$.
In general, $\operatorname{diag}(u, v) \sim \mathbf{1}_{2n}$ only implies that $\operatorname{diag}(u, \mathbf{1}_m) \sim_h \operatorname{diag}(v, \mathbf{1}_m)$ for some $m \in \mathbb{N}$, but this case can be reduced to the previous one by stabilising appropriately, as before.

Commutativity of (52)*:* Let $x \in K_1(A)$ and represent $x = [u]$ with $u \in \mathcal{U}_n^+(A)$. Let moreover $(w_t)_{t\in[0,1]}$ be a homotopy with $w_1 = \operatorname{diag}(u, u^*)$ and $w_0 = \mathbf{1}_{2n}$, so that $\eta_A(x) = [f] - [\mathbf{1}_n]$ with $f(t) := w_t \operatorname{diag}(\mathbf{1}_n, 0)w_t^*$. Then $K_1(\Phi)(x) = [\Phi^+(u)]$ and $(\Phi^+(w_t))_{t\in[0,1]}$ is a homotopy between $\operatorname{diag}(\Phi^+(u), \Phi^+(u)^*)$ and $\mathbf{1}_{2n}$. Therefore

$$\eta_B([\Phi^+(u)]) = \left[\Phi^+(w_t) \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \Phi^+(w_t)^*\right] - [\mathbf{1}_n] = \left[\left(\Phi^+(f)\right)\right] - [\mathbf{1}_n]$$

hence

$$\begin{aligned} \eta_B \circ K_1(\Phi)(x) &= \eta_B([\Phi^+(u)]) \\ &= \left[\left(\Phi^+(f)\right)\right] - [\mathbf{1}_n] \\ &= K_0(S\Phi)([f] - [\mathbf{1}_n]) \\ &= K_0(S\Phi) \circ \eta_A(x), \end{aligned}$$

as desired. □

Example 5.8 (The Bott element, again)**.** The function $\overline{z} \in S\mathbb{C}^+ = C(\mathbb{T})$ is unitary and hence defines an element of $K_1(S\mathbb{C})$. We calculate the image of the class $[\overline{z}]$ under $\eta_{S\mathbb{C}} : K_1(S\mathbb{C}) \to K_0(S^2\mathbb{C})$. To this end, we need a path w of unitaries with $w(0) = \mathbf{1}_2$, $w(1) = \mathrm{diag}(\overline{z}, z)$. The standard construction, exhibited in the proof of Prop. 2.6(d), is

$$w(t) = \begin{pmatrix} \overline{z} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & \sin\left(\frac{\pi t}{2}\right) \\ -\sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix},$$

which is just u_{Bott}, defined in (45). Hence

$$\eta([\overline{z}]) = \left[u_{\mathrm{Bott}} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} u_{\mathrm{Bott}}^* \right] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \right] = \mathfrak{b},$$

the Bott element.

5.3 The index map

Definition 5.9 (Index map)**.** Let A be a C^*-algebra and $J \subset A$ a closed ideal. Let δ be the boundary map to the short exact sequence (25). The *index map* is the group homomorphism $\mathrm{Ind} : K_1(A/J) \longrightarrow K_0(J)$ making the diagram

$$\begin{tikzcd} K_1(A/J) \arrow[r, "\eta_{A/J}"] \arrow[dr, dashed, bend right, "\mathrm{Ind}"'] & K_0(S(A/J)) \arrow[d, "\delta"] \\ & K_0(J) \end{tikzcd}$$

commutative.

Proposition 5.10 (Formula for the index map)**.** Let A be a C^*-algebra and $J \subset A$ a closed ideal. Given $u \in \mathcal{U}_n^+(A/J)$, choose $\widetilde{w} \in \mathcal{U}_{2n}^+(A)$ with $\pi^+(\widetilde{w}) = \mathrm{diag}(u, u^*)$. Then

$$\mathrm{Ind}([u]) = \left[\widetilde{w} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \widetilde{w}^* \right] - \left[\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(J). \tag{56}$$

Proof. We remark first of all that such a lift $\widetilde{w}$ exists by Corollary 3.4). Observe that $\mathrm{Ind}([u])$ does not depend on the choice of lift. Namely, if $\widetilde{w}'$ is another left, then we set $\widetilde{v} = \widetilde{w}'\widetilde{w}^*$ and observe that $\pi^+(v) = \mathbf{1}_{2n}$, hence $\widetilde{v} \in K_0(J^+)$. Therefore, in $\mathcal{V}(J^+)$, we have

$$\left[\widetilde{w}' \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} (\widetilde{w}')^*\right] = \left[\widetilde{v}\widetilde{w} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \widetilde{w}^*\widetilde{v}^*\right] = \left[\widetilde{w} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*\right]$$

We may therefore assume that $\widetilde{w} \sim_h \mathbf{1}_{2n}$ (such a lift exists by Corollary 3.4) . In this case, let $(\widetilde{w}_t)_{t\in[0,1]}$ be a continuous family of unitaries $\widetilde{w}_t \in \mathcal{U}^+_{2n}(A)$ such that $\widetilde{w}_0 = \mathbf{1}_{2n}$, $\widetilde{w}_1 = \widetilde{w}$. Set moreover $w_t = \pi^+(\widetilde{w}_t)$ for $t \in [0,1]$. Then by the definition of η,

$$\eta_{A/J}([u]) = [f] - [\mathbf{1}_n], \qquad f(t) = w_t \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w_t^*.$$

Moreover, the path $\widetilde{f}$ of projections in $M_{2n}(A^+)$ given by $\widetilde{f}(t) = \widetilde{w}_t\,\mathrm{diag}(\mathbf{1}_n, 0)\widetilde{w}_t^*$ is a lift of f, hence by the formula (37), we have

$$\delta([f]) = [\widetilde{f}(1)] - [\widetilde{f}(0)].$$

But this is precisely (56). ☐

Proposition 5.11. Let A be a unital C^*-algebra and let $J \subset A$ be a closed ideal. Let $v \in A$ be a partial isometry such that $\mathbf{1} - v^*v \in J$ and $\mathbf{1} - vv^* \in J$. Then $\pi(v) \in \mathcal{U}(A/J)$ and

$$\mathrm{Ind}([\pi(v)]) = [\mathbf{1} - v^*v] - [\mathbf{1} - vv^*] \in K_0(J).$$

Here we use Remark 5.3 to identify $\mathcal{U}(A/J)$ with $\mathcal{U}^+(A/J)$, in order to see how $\pi(v)$ defines an element of $K_1(A/J)$.

Proof. Since $\mathbf{1} - v^*v, \mathbf{1} - vv^* \in J$, we have

$$\pi(v)^*\pi(v) = \pi(v^*v) = \mathbf{1} \qquad \text{and} \qquad \pi(v)\pi(v)^* = \mathbf{1}.$$

Hence $\pi(v)$ is indeed unitary in A/J. Moreover, one easily checks that

$$u := \begin{pmatrix} v & \mathbf{1} - vv^* \\ \mathbf{1} - v^*v & v^* \end{pmatrix} \in M_2(A)$$

is a unitary lift of $\mathrm{diag}(\pi(v), \pi(v)^*) \in M_2(A/J)$. Now by (56), we have

$$\begin{aligned} \mathrm{Ind}([\pi(v)]) &= \left[u \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} u^*\right] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} vv^* & 0 \\ 0 & \mathbf{1} - v^*v \end{pmatrix}\right] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= [vv^*] + [\mathbf{1} - v^*v] - [\mathbf{1}] \\ &= [\mathbf{1} - v^*v] - [\mathbf{1} - vv^*], \end{aligned}$$

where we used that $[\mathbf{1}] = [vv^*] + [\mathbf{1} - vv^*]$ (see e.g. the proof of Prop. 2.19(a)). ☐

Example 5.12 (The Bott element, once more)**.** Because the adjoint shift operator $S^* \in \mathcal{T}_0^+ = \mathcal{T}$ is a partial isometry lifting the unitary $\bar{z} \in S\mathbb{C}^+$, we have by Prop. 5.11

$$\mathrm{Ind}([\bar{z}]) = [\mathbf{1} - SS^*] - [\mathbf{1} - S^*S] = [\mathbf{1} - SS^*].$$

This gives another proof that $\eta_{S\mathbb{C}}([\bar{z}])$ is a Bott element.

Remember that an operator $T \in \mathbb{B}$ is called *Fredholm* if $\pi(T) \in \mathbb{B}/\mathbb{K}$ is invertible. In this case, its *index* is defined as

$$\mathrm{ind}(T) := \dim\ker(T) - \dim\mathrm{coker}(T) = \dim\ker(T) - \dim\ker(T^*).$$

Remark 5.13. Let $T \in \mathbb{B}$ be a Fredholm operator. That $\pi(T)$ is invertible means that there exists a *parametrix* $S \in \mathbb{B}$ with $TS - \mathbf{1} =: K \in \mathbb{K}$ and $ST - \mathbf{1} =: L \in \mathbb{K}$. Hence $\ker(T) \subseteq \ker(ST) = \ker(\mathbf{1} + L)$, which is the eigenspace to eigenvalue -1 of L. But since L is compact, this is finite-dimensional. Similarly, $\ker(T^*) \subseteq \ker(S^*T^*) = \ker(\mathbf{1} + K^*)$ is finite-dimensional. Thus $\mathrm{ind}(T)$ is well-defined.

Proposition 5.14. Let $V \in \mathbb{B}$ be partial isometry which is Fredholm. Then

$$\mathrm{Ind}([\pi(V)]) = [P_{\ker(V)}] - [P_{\ker(V^*)}] \tag{57}$$

where $P_{\ker(V)}$ and $P_{\ker(V^*)}$ are the orthogonal projections onto $\ker(V)$, respectively $\ker(V^*)$ and Ind is the index map for the pair $\mathbb{K} \subset \mathbb{B}$. In particular,

$$\tau(\mathrm{Ind}(\pi(V))) = \mathrm{ind}(V), \tag{58}$$

where $\tau : K_0(\mathbb{K}) \to \mathbb{Z}$ is the isomorphism that sends a rank one projection to $1 \in \mathbb{Z}$.

Proof. We claim that

$$\mathbf{1} - V^*V = P_{\ker(V)}, \qquad \text{and} \qquad \mathbf{1} - VV^* = P_{\ker(V^*)}. \tag{59}$$

Indeed, if $\varphi \in \ker(V)$, then $(\mathbf{1} - V^*V)\varphi = \varphi$, while if $\varphi \in \ker(V)^\perp = \mathrm{im}(V^*)$, we have $\varphi = V^*\psi$ for some $\psi \in H$. Therefore using (12)

$$(\mathbf{1} - V^*V)\varphi = \varphi - V^*VV^*\psi = \varphi - V^*\psi = \varphi - \varphi = 0.$$

This shows the first identity in (59); the second follows from replacing V by V^*. Formula (57) is now a consequence of Prop. 5.11.
Formula (58) follows from observing that $\tau([P] - [Q]) = \mathrm{tr}(P) - \mathrm{tr}(Q)$ (see Example 2.20) and that the trace of a projection is equal to its rank. □

5.4 The exponential map

We start by deriving a more explicit formula for the Bott map in terms of the K_1 group.

Proposition 5.15. Let A be a C^*-algebra. For a projection $p \in M_n(A^+)$, define the *projection loop* $f_p \in M_n(C([0,1], A^+))$ by

$$f_p(t) = e^{-2\pi i t} p + \mathbf{1}_n - p. \tag{60}$$

Then the composition $\beta'_A := \eta_{SA}^{-1} \circ \beta_A : K_0(A) \to K_1(SA)$ is given by the formula

$$\beta'_A([p] - [q]) \longmapsto [f_p f_q^*].$$

for projections $p, q \in M_n(A^+)$ such that $p - q \in M_n(A)$.

Remark 5.16. We generally do *not* have $f_p \in \mathcal{U}_n^+(SA)$. However, if $[p] - [q] \in K_0(A)$ such that $p - q \in M_n(A)$, then $\varepsilon_A(p) = \varepsilon_A(q)$. Since $\varepsilon_A(f_p(t)) = f_{\varepsilon_A(p)}(t) \in M_n(\mathbb{C})$, we therefore obtain $\varepsilon_A(f_q(t) f_q(t)^*) = \mathbf{1}_n$ in $M_n(\mathbb{C})$, hence $f_p f_q^* \in \mathcal{U}_n^+(SA)$.

Proof. It suffices to verify this for unital algebras A, since $\eta_{SA}^{-1} \circ \beta_A$ is the restriction of $\eta_{SA^+}^{-1} \circ \beta_{A^+}$. As $\eta_{SA}^{-1} \circ \beta_A$ is homomorphism, it moreover suffices to verify that

$$\eta_{SA}([f_p]) = \mathfrak{b} \times [p] = [p_{\mathrm{Bott}} \otimes p] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \otimes p \right].$$

To calculate the left hand side, we need to choose a path w in $\mathcal{U}_{2n}^+(SA)$ connecting $\mathbf{1}_{2n}$ to $\mathrm{diag}(f_p, f_p^*)$; in other words, an element $w \in C([0,1]^2, A)$ with $w(t,0) = w(0,s) = w(1,s) = \mathbf{1}_{2n}$ and $w(t,1) = \mathrm{diag}(f_p, f_p^*)$. Then

$$\eta_{SA}([f_p]) = \left[w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^* \right] - \left[\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \right]$$

A possible choice is

$$\begin{aligned} w &= \begin{pmatrix} f_p & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi s}{2}\right) & -\sin\left(\frac{\pi s}{2}\right) \\ \sin\left(\frac{\pi s}{2}\right) & \cos\left(\frac{\pi s}{2}\right) \end{pmatrix} \begin{pmatrix} f_p^* & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi s}{2}\right) & \sin\left(\frac{\pi s}{2}\right) \\ -\sin\left(\frac{\pi s}{2}\right) & \cos\left(\frac{\pi s}{2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{\pi s}{2}\right) p & -\sin\left(\frac{\pi s}{2}\right) \bar{z} p \\ \sin\left(\frac{\pi s}{2}\right) z p & \cos\left(\frac{\pi s}{2}\right) p \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi s}{2}\right) & \sin\left(\frac{\pi s}{2}\right) \\ -\sin\left(\frac{\pi s}{2}\right) & \cos\left(\frac{\pi s}{2}\right) \end{pmatrix} + \begin{pmatrix} \mathbf{1}_n - p & 0 \\ 0 & \mathbf{1}_n - p \end{pmatrix} \\ &= u_{\mathrm{Bott}} \otimes p + \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \otimes (\mathbf{1}_n - p), \end{aligned}$$

where we wrote $z = e^{2\pi i t}$ and u_{Bott} is the unitary (45) used in the definition (46) of the

Bott projection. Therefore

$$\begin{aligned}\eta_{SA}([f_p]) &= \left[u_{\mathrm{Bott}} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} u_{\mathrm{Bott}}^* \otimes p\right] + \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathbf{1}_n - p)\right] - \left[\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= [p_{\mathrm{Bott}} \otimes p] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \otimes p\right],\end{aligned}$$

which is what needed to be shown. □

Definition 5.17 (Exponential map)**.** Let A be a C^*-algebra and $J \subset A$ be a closed ideal. Then the corresponding *exponential map* is the map $\mathrm{Exp} : K_0(A/J) \to K_1(J)$ such that the square

$$\begin{tikzcd} K_0(A/J) \arrow[r, "\beta_{A/J}", "\cong"'] \arrow[d, dashed, "\mathrm{Exp}"'] & K_0(S^2(A/J)) \arrow[d, "S\delta"] \\ K_1(J) \arrow[r, "\eta_J", "\cong"'] & K_0(SJ), \end{tikzcd} \tag{61}$$

commutes, where $S\delta$ is the boundary map to the suspended ideal $SJ \subseteq SA$.

Proposition 5.18 (Formula for the exponential map)**.** Let A be a C^*-algebra and let $J \subset A$ be a closed ideal. The exponential map can be described as follows. Given $x \in K_0(A/J)$, represent $x = [p] - [\mathbf{1}_k]$ with a projection $p \in M_n((A/J)^+)$, $n \geq k$ such that $p - \mathbf{1}_k \in M_n(A/J)$. Then

$$\mathrm{Exp}(x) = \big[\exp(2\pi i \widetilde{p})\big], \tag{62}$$

where $\widetilde{p} \in M_n(A^+)$ is some self-adjoint lift of p.

Remark 5.19. We emphasize that $\widetilde{p}$ is not required to be a projection as well. In fact, if $\widetilde{p} \in M_n(A^+)$ is also a projection, then it has spectrum $\sigma(\widetilde{p}) \subseteq \{0, 1\}$, hence $\exp(2\pi i\widetilde{p}) = \mathbf{1}_n$, which represents the zero element of $K_1(J)$. So in this sense, the exponential map provides a measure of the failure of p to lift to a projection.

Proof. Observe first that indeed $\exp(-2\pi i\widetilde{p}) \in \mathcal{U}_m^+(J)$, as it is a unitary in $M_m(J^+)$ and

$$\pi^+\big(\exp(2\pi i\widetilde{p})\big) = \exp(2\pi i p) = \mathbf{1}_m,$$

hence $\exp(2\pi i\widetilde{p}) - \mathbf{1}_m \in M_m(J)$.
As before, we may assume that A is unitary. Let $p \in M_n(A/J)$ be a projection. To use

the previous results, extend the Diagram (61) as follows:

$$\begin{tikzcd}
 & K_1(S(A/J)) \arrow[d, "\eta_{S(A/J)}"] \arrow[dd, bend left=60, "S\,\mathrm{Ind}"] \\
K_0(A/J) \arrow[ur, bend left, "\beta'_{A/J}"] \arrow[r, "\beta_{A/J}", "\cong"'] \arrow[d, dashed, "\mathrm{Exp}"'] & K_0(S^2(A/J)) \arrow[d, "S\delta"] \\
K_1(J) \arrow[r, "\eta_J", "\cong"'] & K_0(SJ),
\end{tikzcd}$$

where $S\,\mathrm{Ind} : K_1(S(A/J)) \to K_0(SJ)$ is the index map corresponding to the suspended ideal $SJ \subset SA$. Then by Prop. 5.15 and Def. 5.9, we can write

$$(S\delta \circ \beta_{A/J})([p]) = (S\delta \circ \eta_{S(A/J)})([f_p]) = S\,\mathrm{Ind}([f_p]),$$

We then want to verify

$$S\,\mathrm{Ind}([f_p]) = \eta_J([p]).$$

To calculate the left hand side, let $\widetilde{w} \in M_{2n}(SA)$ be a unitary lift of $\mathrm{diag}(f_p, f_p^*)$, that is $\widetilde{w}(0) = \widetilde{w}(1) = \mathbf{1}_{2n}$ and $\pi^+(\widetilde{w}(t)) = \mathrm{diag}(f_p(t), f_p^*(t))$ for all $t \in [0,1]$. Then by the formula for the index map, Prop. 5.10, we have

$$S\,\mathrm{Ind}([f_p]) = \left[\widetilde{w}\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\widetilde{w}^*\right] - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right].$$

On the other hand, let $\widetilde{p} \in M_n(A)$ be a self-adjoint lift of p and define $\widetilde{u}(t) := \exp(2\pi i t\widetilde{p})$. Then

$$\pi^+(\widetilde{u}(t)) = \exp(2\pi i t p) = e^{2\pi i t}p + \mathbf{1}_n - p = f_p(t)^*.$$

Therefore, if we set

$$\widetilde{v}(t) := \widetilde{w}(t)\begin{pmatrix}\widetilde{u}(t) & 0\\ 0 & \widetilde{u}(t)^*\end{pmatrix},$$

then $\pi^+(\widetilde{v}(t)) = \mathbf{1}_{2n}$ for all $t \in [0,1]$, hence we obtain a continuous path of unitaries in $M_{2n}(J^+)$ with $\widetilde{v}(0) = \mathbf{1}_{2n}$ and $\widetilde{v}(1) = \mathrm{diag}(\exp(2\pi i\widetilde{p}), \exp(-2\pi i\widetilde{p}))$. Therefore, by definition of η_J,

$$\begin{aligned}
\eta_J([\exp(2\pi i\widetilde{p})]) &= \left[\widetilde{v}\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\widetilde{v}^*\right] - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right]\\
&= \left[\widetilde{w}\begin{pmatrix}\widetilde{u} & 0\\ 0 & \widetilde{u}^*\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\widetilde{u}^* & 0\\ 0 & \widetilde{u}\end{pmatrix}\widetilde{w}^*\right] - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right]\\
&= \left[\widetilde{w}\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\widetilde{w}^*\right] - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right] = S\,\mathrm{Ind}([p])
\end{aligned}$$

This finishes the proof. □

5.5 The six-term exact sequence

Theorem 5.20 (The six-term sequence)**.** Let A be a C^*-algebra and let $J \subset A$ be a closed ideal. Then the six-term sequence

$$\begin{tikzcd}
K_0(J) \arrow[r, "K_0(\iota)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(A/J) \arrow[d, "\mathrm{Exp}"] \\
K_1(A/J) \arrow[u, "\mathrm{Ind}"] & K_1(A) \arrow[l, "K_1(\pi)"] & K_1(J) \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

is exact.

Proof. So far, from Thm. 3.10, we know the exactness of the (non-dashed) spiral sequence

$$\begin{tikzcd}
K_0(S^2J) \arrow[r, "K_0(S^2\iota)"] & K_0(S^2A) \arrow[r, "K_0(S^2\pi)"] & K_0(S^2(A/J)) \arrow[dd, bend left, "S\delta"] \\
K_0(J) \arrow[r, "K_0(\iota)"] \arrow[u, dashed, "\beta_J", "\cong"'] & K_0(A) \arrow[r, "K_0(\pi)"] \arrow[u, dashed, "\beta_A", "\cong"'] & K_0(A/J) \arrow[u, dashed, "\beta_{A/J}", "\cong"'] \\
K_0(S(A/J)) \arrow[u, "\delta"] & K_0(SA) \arrow[l, "K_0(S\pi)"] & K_0(SJ). \arrow[l, "K_0(S\iota)"]
\end{tikzcd}$$

However, the Bott periodicity isomorphisms (dashed) provide an exact K_0-K_0S-six-term sequence, where the right boundary map $K_0(A/J) \to K_0(SJ)$ is $S\delta \circ \beta_{A/J}$. Here the commutativity of the above diagram follows from naturality of β. The commutativity of the diagram

$$
\begin{tikzcd}
K_0(J) \arrow[r, "K_0(\iota)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(A/J) \arrow[d, "S\delta\circ\beta_{A/J}"] \arrow[dd, dashed, bend left=60, "\mathrm{Exp}"] \\
K_0(S(A/J)) \arrow[u, "\delta"] & K_0(SA) \arrow[l, "K_0(S\pi)"'] & K_0(SJ) \arrow[l, "K_0(S\iota)"'] \\
K_1(A/J) \arrow[u, dashed, "\eta_{A/J}"] \arrow[uu, dashed, bend left=60, "\mathrm{Ind}"] & K_1(A) \arrow[l, dashed, "K_1(\pi)"'] \arrow[u, dashed, "\eta_A"] & K_1(J), \arrow[l, "K_1(\iota)"'] \arrow[u, dashed, "\eta_J"]
\end{tikzcd}
$$

which follows from naturality of η, then implies the exactness of the corresponding K_0-K_1-sequence. □

References

[1] N. P. Brown and N. Ozawa. C^**-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[2] J. Dixmier. C^**-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[3] G. Murphy. C^**-algebras and operator theory*. Academic Press Inc., Boston-San Diego-New York, 1990.

[4] N. E. Wegge-Olsen. K*-theory and* C^**-algebras*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. A friendly approach.

[5] D. Werner. *Funktionalanalysis*.