Orbifold completion
of 3-categories

Nils Carqueville

Universität Wien

based on joint work with Ilka Brunner, Catherine Meusburger, Vincentas Mulevičius, Daniel Plencner, Ana Ros Camacho, Ingo Runkel, Gregor Schaumann, Daniel Scherl, and **Lukas Müller**: arXiv:2307.06485 [math.QA]

overview: arXiv:2307.16674 [math-ph]

slides: `https://carqueville.net/nils/3dorb.pdf`

In a nutshell

Orbifold data …are algebraic representations of Pachner moves

…are objects of a higher Morita category

…are special defects in defect TQFT

…are gaugeable (non-invertible) symmetries

…give rise to state sum models

In a nutshell

Orbifold data ... are algebraic representations of Pachner moves
... are objects of a higher Morita category
... are special defects in defect TQFT
... are gaugeable (non-invertible) symmetries
... give rise to state sum models

Theorem. Let $\mathcal{T}$ be 3-category with duals. The higher Morita category $\mathcal{T}_{\mathrm{orb}}$ of orbifold data in $\mathcal{T}$ has duals.

Theorem. Let $\mathcal{Z}$ be 3d defect TQFT and $\mathcal{D}_{\mathcal{Z}}$ its 3-category with duals. From $(\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}}$ one obtains 3d **defect TQFT** $\mathcal{Z}_{\mathrm{orb}}$.

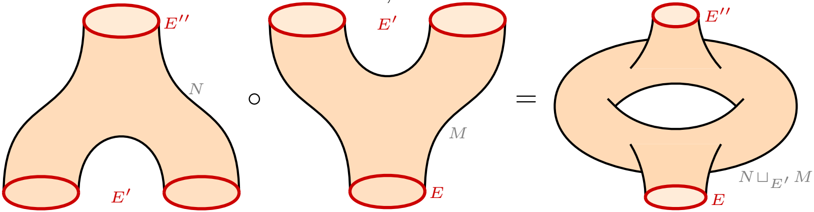
Applications.

- "Defect state sum models are orbifolds of the trivial defect TQFT."
- "Reshetikhin–Turaev defect TQFTs without thinking"
- "Douglas–Reutter 4-manifold invariants via orbifolds"

Closed TQFT

An **n-dimensional closed oriented TQFT** is symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{or}}_{n,n-1} \longrightarrow \mathcal{C}$$



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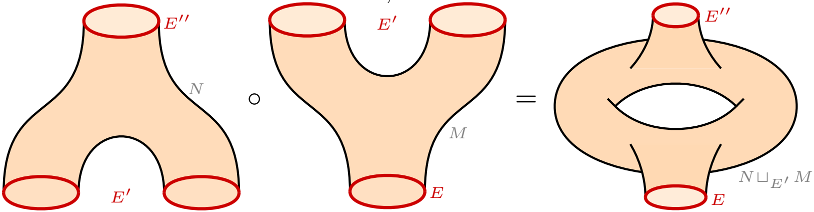


Classification.

(1d closed oriented TQFTs) $\cong$ (dualisable objects)

(2d closed oriented TQFTs) $\cong$ (commutative Frobenius algebras)

(3d closed oriented TQFTs) $\cong$ (J-algebras)

(4d closed oriented TQFTs) $\cong$??

Juhasz 2014

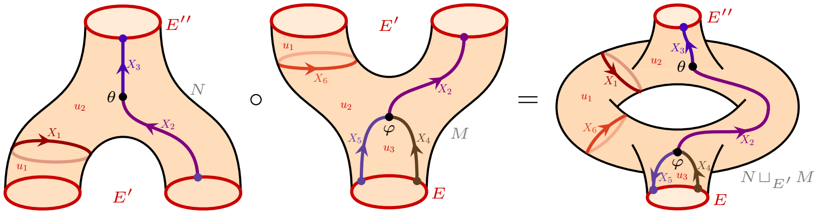
Defect TQFT

An n-**dimensional defect TQFT** is symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \longrightarrow \mathcal{C}$$

depending on set of **defect data** $\mathbb{D}$ consisting of

- set D_n of "bulk theories"
- sets D_j of j-dimensional "defects" for $j \in \{0, 1, \ldots, n-1\}$
- adjacency rules...



Davydov/Kong/Runkel 2011, Carqueville/Runkel/Schaumann 2017

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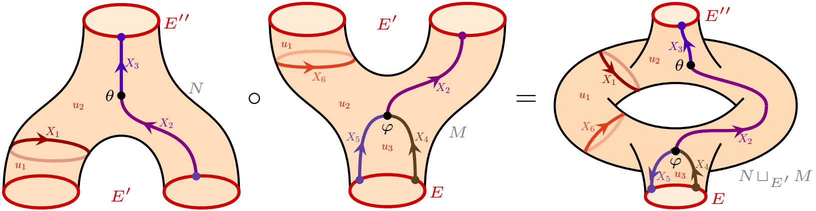


Non-full embedding $\mathrm{Bord}^{\mathrm{or}}_{n,n-1} \hookrightarrow \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D})$ for all $u \in D_n$

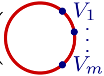
Davydov/Kong/Runkel 2011, Carqueville/Runkel/Schaumann 2017

Examples of 2d defect TQFTs

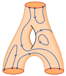
Trivial defect TQFT $\mathcal{Z}_2^{\mathrm{triv}} \colon \mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv2}}) \longrightarrow \mathrm{Vect}_{\mathbb{k}}$

$D_2^{\mathrm{triv2}} := \{\mathbb{k}\}$

$D_1^{\mathrm{triv2}} := \mathrm{Ob}(\mathrm{vect}_{\mathbb{k}})$

$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) := V_1 \otimes \cdots \otimes V_m$

$D_0^{\mathrm{triv2}} := \mathrm{Mor}(\mathrm{vect}_{\mathbb{k}})$

$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) :=$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\mathbb{k}}$)

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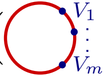
$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) := V_1 \otimes \cdots \otimes V_m$

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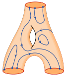
$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) :=$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\Bbbk}$)

State sum models $\mathcal{Z}_2^{\mathrm{ss}}$:
separable symmetric Frobenius $\Bbbk$-algebras and bimodules

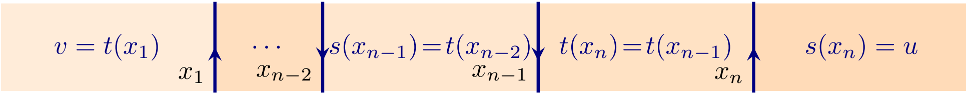
B-twisted sigma models $\mathcal{Z}^{\mathrm{B}\sigma}$:
Calabi–Yau manifolds and their derived categories

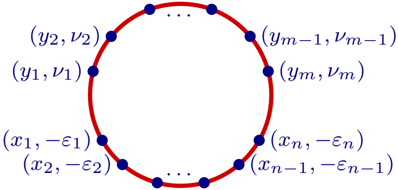
Landau–Ginzburg models $\mathcal{Z}^{\mathrm{LG}}$:
isolated singularities and matrix factorisations

Higher categories from defect TQFTs

Theorem. For $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}$, there is pivotal 2-category $\mathcal{D}_{\mathcal{Z}}$ with

- objects: elements of D_2
- 1-morphisms $X\colon u \longrightarrow v$: lists of composable elements of D_1



- $\mathrm{Hom}(X,Y) = \mathcal{Z}\Bigg($  $\Bigg)$

- composition: "pair-of-pants with defects"

Davydov/Kong/Runkel 2011

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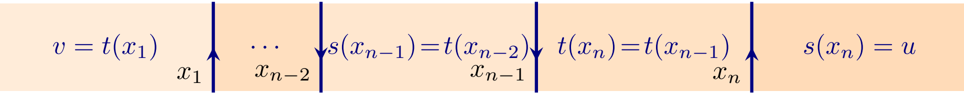
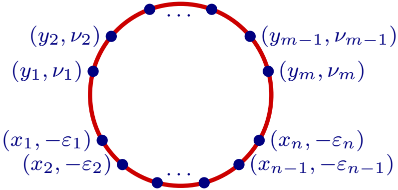


- $\mathrm{Hom}(X,Y) = \mathcal{Z}\Bigg($  $\Bigg)$

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Examples.

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{triv}}} \cong \mathrm{B}\,\mathrm{vect}_{\Bbbk}$$

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{ss}}} \cong \mathrm{ssFrob}(\mathrm{vect}_{\Bbbk}) \cong \left(\mathcal{D}_{\mathcal{Z}_2^{\mathrm{triv}}}\right)_{\mathrm{orb}}^{\odot}$$

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{LG}}} \cong \mathcal{LG}$$

Davydov/Kong/Runkel 2011

Examples of 3d defect TQFTs

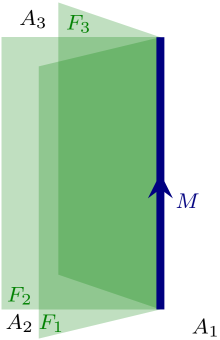
Reshetikhin–Turaev defect TQFT $\mathcal{Z}^{\mathrm{RT}}_{\mathcal{M}}$ for modular fusion category $\mathcal{M}$:

$D^{\mathrm{RT}}_3 := \{$commutative Δ-separable Frobenius algebras A in $\mathcal{M}\}$

$D^{\mathrm{RT}}_2 := \{\Delta$-sep. sym. Frobenius alg. F with comp. bimodule structure$\}$

$D^{\mathrm{RT}}_1 := \{$multimodules $M\}$

$D^{\mathrm{RT}}_0 := \{$multimodule maps$\}$



Trivial defect TQFT $\mathcal{Z}^{\mathrm{triv}}_3 \;\cong\; \mathcal{Z}^{\mathrm{RT}}_{\mathrm{vect}_{\Bbbk}}\Big|_{D^{\mathrm{RT}}_3 \longrightarrow \{\Bbbk\}}$

Kapustin/Saulina 2010, Carqueville/Runkel/Schaumann 2017, Koppen/Mulevičius/Runkel/Schweigert 2021, Carqueville/Müller 2023

Higher categories from defect TQFTs

Theorem. For $\mathcal{Z}\colon \operatorname{Bord}^{\text{def}}_{3,2}(\mathbb{D}) \longrightarrow \mathcal{C}$, there is 3-category with duals $\mathcal{D}_{\mathcal{Z}}$:

- objects: elements of D_3
- k-morphisms: $(3-k)$-fold cylinders over defect k-balls, $k \in \{1, 2\}$
- 3-morphisms: $\mathcal{Z}($"defect 2-sphere"$)$
- composition: "pair-of-pants with defects"
- duals: bending lines and surfaces

Carqueville/Meusburger/Schaumann 2016, Barrett/Meusburger/Schaumann 2012

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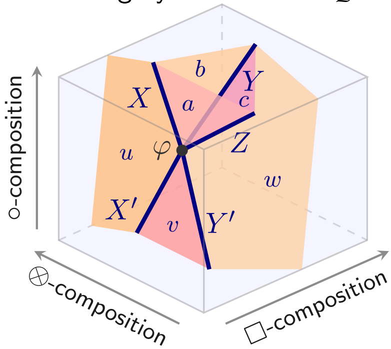

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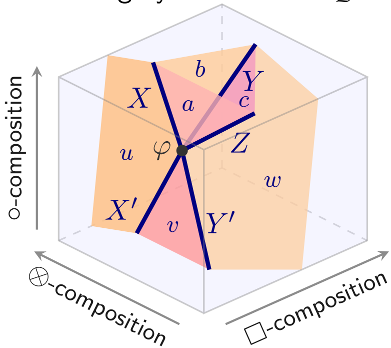

Examples.

$$\mathcal{D}_{\mathcal{Z}_3^{\mathrm{triv}}} \cong \mathrm{B}\,\mathrm{ssFrob}(\mathrm{vect}_{\Bbbk}) \cong \mathrm{B}\,\mathcal{D}_{\mathcal{Z}_2^{\mathrm{ss}}}$$

$$\mathcal{D}_{\mathcal{Z}_3^{\mathrm{ss}}} \cong \left(\mathcal{D}_{\mathcal{Z}_3^{\mathrm{triv}}}\right)^{\odot}_{\mathrm{orb}} \supset \mathrm{sFus}_{\Bbbk}$$

$$\mathcal{D}_{\mathcal{Z}_{\mathcal{M}}^{\mathrm{RT}}} \cong \left(\mathrm{B}\,\Delta\mathrm{ssFrob}(\mathcal{M})\right)\right)_{\mathrm{orb}}$$

Carqueville/Meusburger/Schaumann 2016, Barrett/Meusburger/Schaumann 2012, Carqueville/Müller 2023

Examples of n-dimensional defect TQFTs

Euler defect TQFT $\mathcal{Z}^{\mathrm{eu}}_{\Psi} \colon \mathrm{Bord}^{\mathrm{def}}_{n,n-1} \longrightarrow \mathrm{Vect}_{\Bbbk}$, where

$\mathrm{Bord}^{\mathrm{def}}_{n,n-1}$: stratified bordisms without labels
$\Psi = (\psi_1, \dots, \psi_n) \in (\Bbbk^{\times})^n$

$$\mathcal{Z}^{\mathrm{eu}}_{\Psi}(\text{object } E) := \Bbbk$$

$$\mathcal{Z}^{\mathrm{eu}}_{\Psi}(\text{bordism } M) := \prod_{j=1}^{n} \prod_{j\text{-strata}\, \sigma_j \subset M} \psi_j^{\chi(\sigma_j) - \frac{1}{2}\chi(\partial \sigma_j)}$$

Quinn 1995, Carqueville/Runkel/Schaumann 2017

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Euler completion $\mathcal{Z}^{\odot}$ of any defect TQFT $\mathcal{Z}$ satisfies

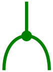
$$\big(\mathcal{Z}^{\odot}\big)^{\odot} \cong \mathcal{Z}^{\odot} \qquad \mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi} \cong \mathcal{Z}^{\odot}$$

Euler completion $\mathcal{D}^{\odot}_{\mathcal{Z}} \cong \mathcal{D}_{\mathcal{Z}^{\odot}}$ of higher defect categories

Quinn 1995, Carqueville/Runkel/Schaumann 2017

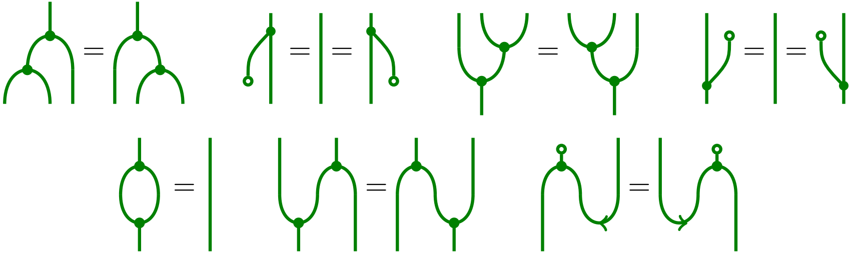
Δ-separable symmetric Frobenius algebras

$A \in \mathcal{C}$ with

$\mu =$  $: A \otimes A \longrightarrow A$  $: \mathbb{1} \longrightarrow A$

$\Delta =$  $: A \longrightarrow A \otimes A$  $: A \longrightarrow \mathbb{1}$

such that



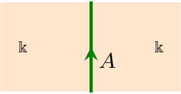
(A need *not* be commutative.)

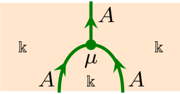
State sum models

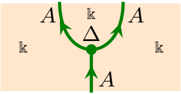
<u>*Input*</u>: Δ-separable symmetric Frobenius $\Bbbk$-algebra (A, μ, Δ)

(1) Choose oriented **triangulation** t for every bordism Σ in $\mathrm{Bord}_{2,1}^{\mathrm{or}}$

(2) **Decorate Poincaré dual** graph with $(\Bbbk, A, \mu, \Delta)$:







(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\boxed{\mathcal{Z}_A(\Sigma) = \mathcal{Z}_2^{\mathrm{triv}}(\Sigma^{t,A})}$

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Input: Δ-separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)

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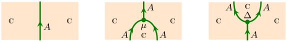

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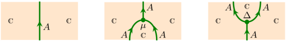

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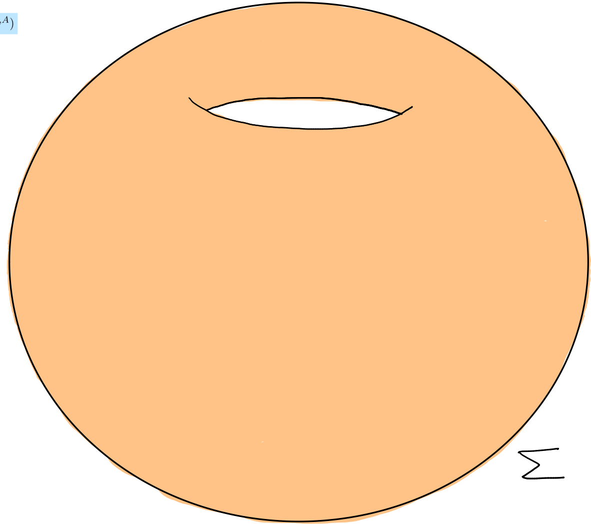

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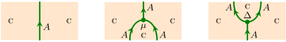

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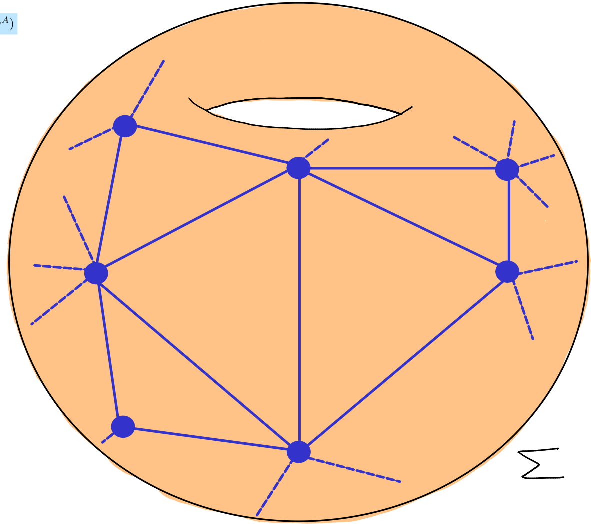

Σ

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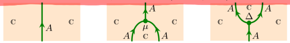

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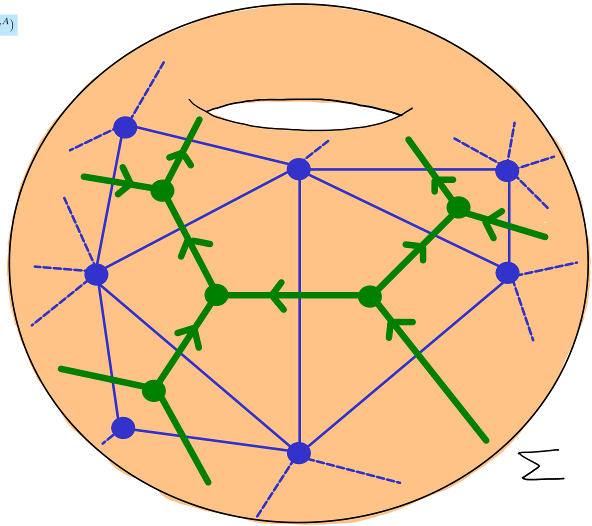

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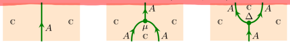

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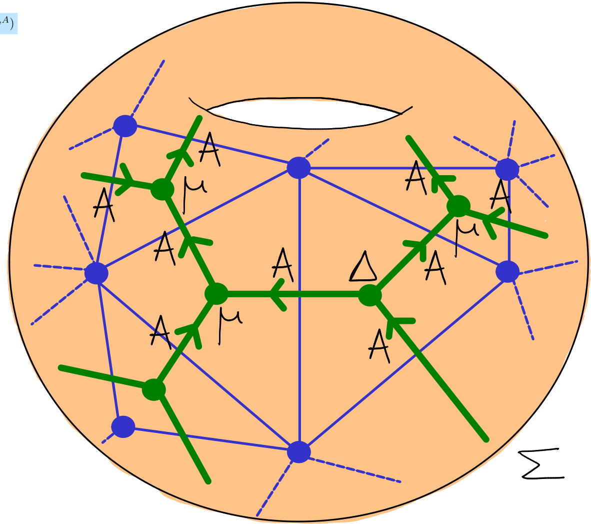

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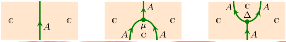

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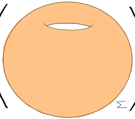
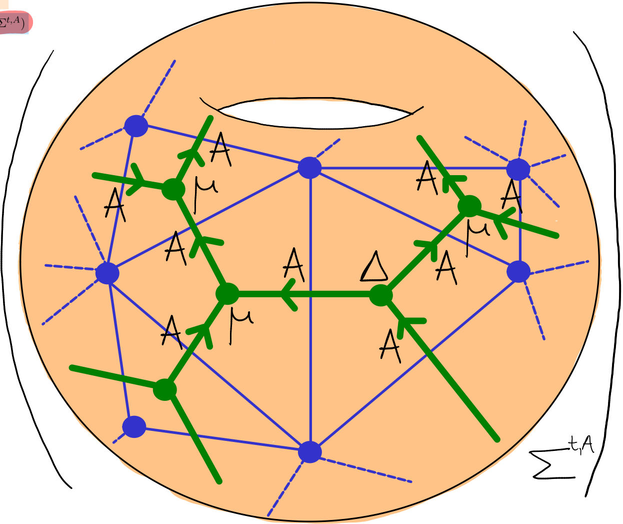
$\mathcal{Z}_A^{ss}\Big($  $\Big) \overset{def}{=} \mathcal{Z}^{triv}\Big($  $\Big)$

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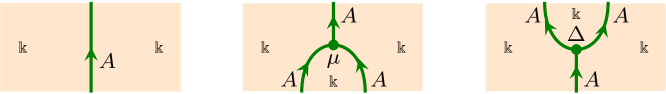

(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\boxed{\mathcal{Z}_A(\Sigma) = \mathcal{Z}_2^{\mathrm{triv}}(\Sigma^{t,A})}$

Theorem. Construction yields TQFT $\mathcal{Z}_A \colon \mathrm{Bord}_{2,1}^{\mathrm{or}} \longrightarrow \mathrm{Vect}_{\mathbb{k}}$.

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models

Input: Δ-separable symmetric Frobenius $\Bbbk$-algebra (A, μ, Δ)

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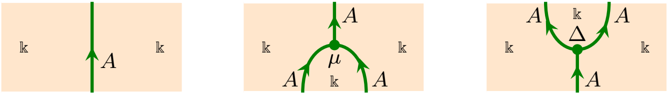

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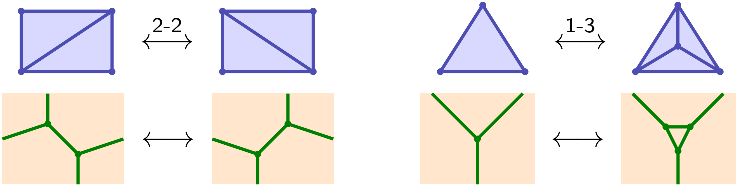
Proof sketch: Defining properties of (A, μ, Δ) encode invariance under **Pachner moves** $\Longrightarrow$ independent of choice of triangulation:



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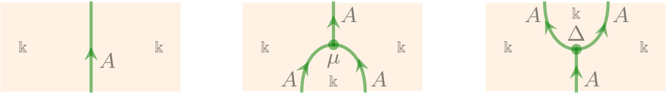

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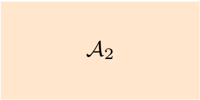
Theorem. Construction yields TQFT $\mathcal{Z}_A \colon \mathrm{Bord}_{2,1}^{\mathrm{or}} \longrightarrow \mathrm{Vect}_{\Bbbk}$.

No need to consider only algebras over $\Bbbk$!

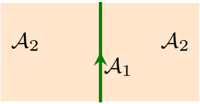
Orbifolds

Definition. Let $\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathcal{C}$ be defect TQFT.

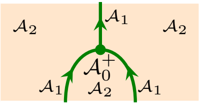
An **orbifold datum** for $\mathcal{Z}$ is $\mathcal{A} \equiv (\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0^+, \mathcal{A}_0^-)$:



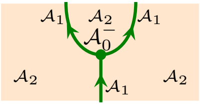
$\mathcal{A}_2 \in D_2$



$\mathcal{A}_1 \in D_1$

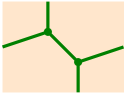
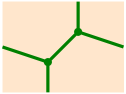
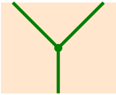
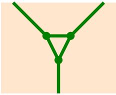


$\mathcal{A}_0^+ \in D_0$



$\mathcal{A}_0^- \in D_0$

such that *Pachner moves become identities* under $\mathcal{Z}$:

$\mathcal{Z}\Big($  $\Big) \overset{!}{=} \mathcal{Z}\Big($  $\Big)$ $\qquad$ $\mathcal{Z}\Big($  $\Big) \overset{!}{=} \mathcal{Z}\Big($  $\Big)$

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

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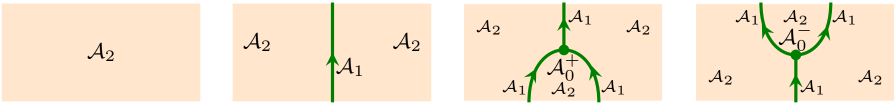

$\mathcal{A}_2 \in D_2$ $\quad$ $\mathcal{A}_1 \in D_1$ $\quad$ $\mathcal{A}_0^+ \in D_0$ $\quad$ $\mathcal{A}_0^- \in D_0$

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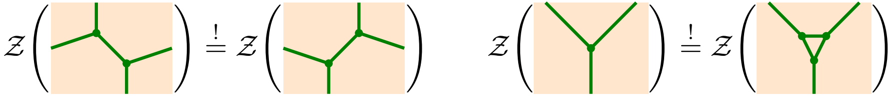

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathcal{C}$$

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

Algebraic characterisation of orbifolds

Theorem.
2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{D}_{\mathcal{Z}}$

Lemma.
$\{$orbifold data for $\mathcal{Z}\} \cong \{\Delta$-separable symmetric Frobenius algebras in $\mathcal{D}_{\mathcal{Z}}\}$

Davydov/Kong/Runkel 2011

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Examples.

- Δ-separable symmetric Frobenius algebras in $\mathrm{BVect}_{\Bbbk}$
  $= \Delta$-separable symmetric Frobenius $\Bbbk$-algebras ☺
  $\implies$ $\boxed{\mathcal{Z}_A = (\mathcal{Z}^{\mathrm{triv}})_A}$ (*"State sum models are orbifolds of the trivial TQFT."*)
- A ***G*-action** in $\mathcal{D}_{\mathcal{Z}}$ is 2-functor $\rho \colon \mathrm{B}\underline{G} \longrightarrow \mathcal{D}_{\mathcal{Z}}$.

  Lemma. $A_G := \bigoplus_{g \in G} \rho(g)$ is Δ-separable Frobenius algebra in $\mathcal{D}_{\mathcal{Z}}$.

  $\implies$ G-orbifolds are orbifolds: $\boxed{\mathcal{Z}^G = \mathcal{Z}_{A_G}}$ ☺

Orbifolds unify gauging of symmetry groups and state sum models.

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

Orbifold completion

Orbifold completion of pivotal 2-category $\mathcal{B}$ is pivotal 2-category $\mathcal{B}_{\mathrm{orb}}$:

– objects: **Δ-separable symmetric Frobenius algebras** $A \in \mathcal{B}(\alpha, \alpha)$

– Hom categories = bimodule categories

Theorem. $\mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{orb}} \cong (\mathcal{B}_{\mathrm{orb}})_{\mathrm{orb}}$

Carqueville/Runkel 2012

see also Douglas/Reutter 2018, Gaiotto/Johnson-Freyd 2019

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Theorem & Definition. (**Orbifold equivalence** $\alpha \sim \beta$)
If $X \in \mathcal{B}(\alpha, \beta)$ has *invertible* $\dim(X) \in \mathrm{End}(1_\beta)$, then:

– $A := X^\dagger \otimes X$ is *separable* symmetric Frobenius algebra in $\mathcal{B}(\alpha, \alpha)$

– $X : (\alpha, A) \rightleftarrows (\beta, 1_\beta) : X^\dagger$ is adjoint equivalence in $\mathcal{B}_{\mathrm{orb}}$

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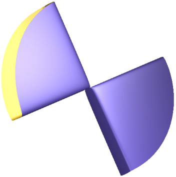
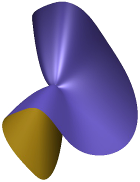
Example. $\mathcal{B} = \mathcal{D}_{\mathcal{Z}^{\mathrm{LG}}} \quad \Longrightarrow$



A_{11}

$\sim$



E_6

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

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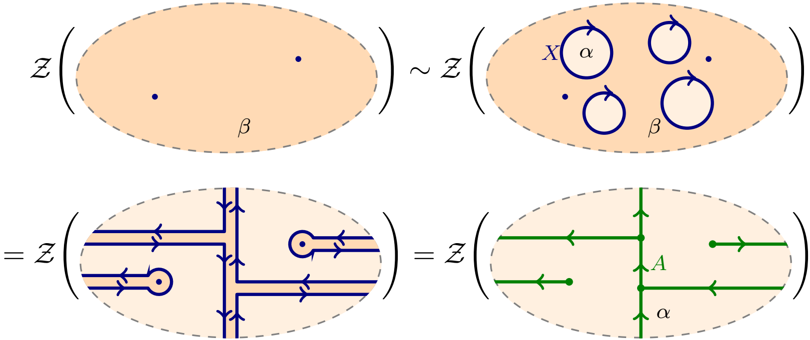

Carqueville/Runkel 2012

Orbifold defect TQFT

Let $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}$ be defect TQFT.

Get new defect data $\mathbb{D}^{\mathrm{orb}}$ with $D_j^{\mathrm{orb}} := \big\{ j\text{-cells of } (\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}} \big\}$.

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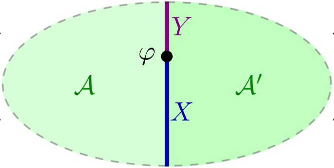
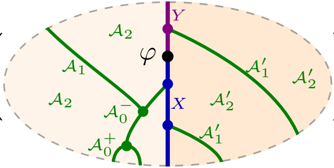
Definition & Theorem.
The **orbifold defect TQFT**

$$\mathcal{Z}_{\mathrm{orb}}\colon \operatorname{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}^{\mathrm{orb}}) \longrightarrow \mathcal{C}$$

is given by substratification:

$\mathcal{Z}_{\mathrm{orb}}\Bigg($ 

$\Bigg) = \mathcal{Z}\Bigg($ 

$\Bigg)$

Carqueville/Runkel 2012

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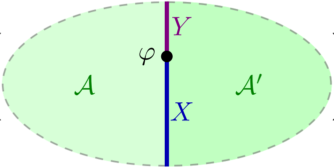
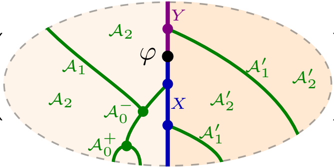


$\Bigg) \; = \; \mathcal{Z}\Bigg($



$\Bigg)$

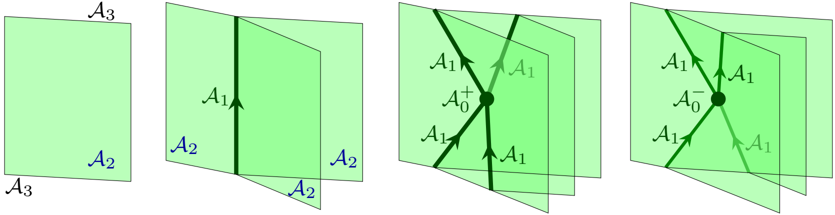
Example. *Defect state sum models are Euler completed orbifolds of the trivial defect TQFT:*

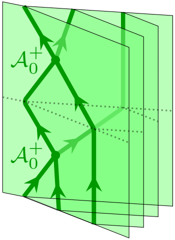
$$\mathcal{Z}_2^{\mathrm{ss}} \;\cong\; \big(\mathcal{Z}_2^{\mathrm{triv}}\big)_{\mathrm{orb}}^{\odot}$$

Carqueville/Runkel 2012, Lauda/Pfeiffer 2005, Davydov/Kong/Runkel 2011, Mulevičius 2022

3-dimensional orbifold data

Let $\mathcal{T}$ be 3-category with duals. An **orbifold datum** $\mathcal{A}$ in $\mathcal{T}$ is





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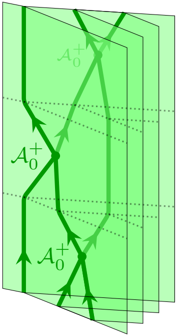
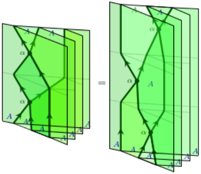
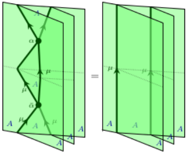


etc.

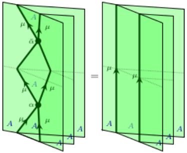
Carqueville/Runkel/Schaumann 2017



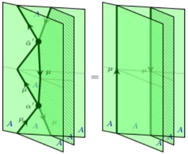
(O1)



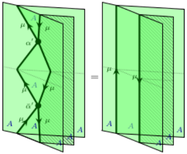
(O2)



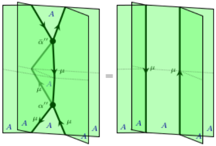
(O3)



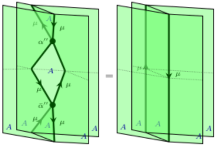
(O4)



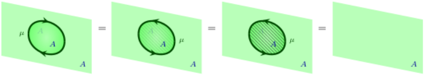
(O5)



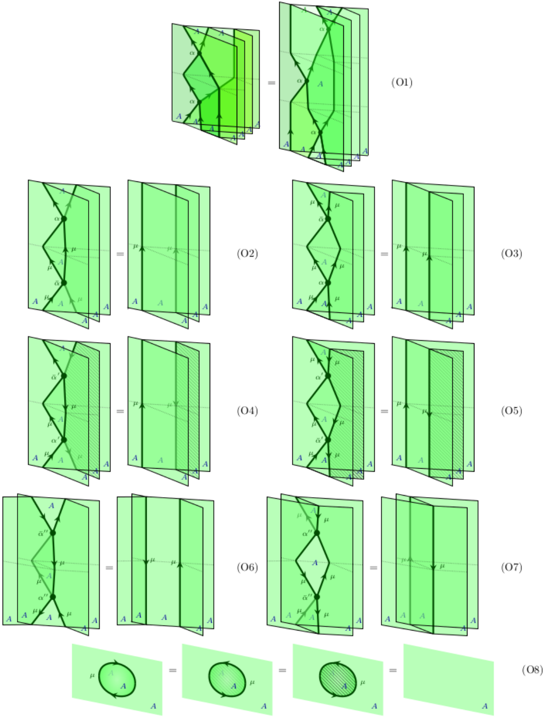
(O6)



(O7)



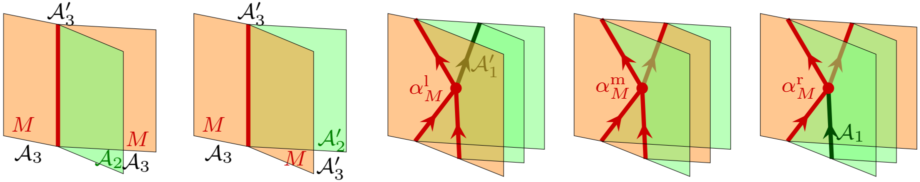
(O8)



Which 3-category $\mathcal{T}_{\mathrm{orb}}$ are orbifold data objects of?

Representations of 3-dimensional orbifold data

Let $\mathcal{A}$ and $\mathcal{A}'$ be orbifold data in $\mathcal{T}$. An **$\mathcal{A}'$-$\mathcal{A}$-bimodule** $\mathcal{M}$ is



subject to pentagon axioms.

Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

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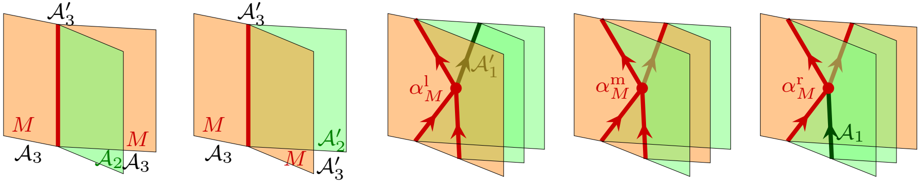

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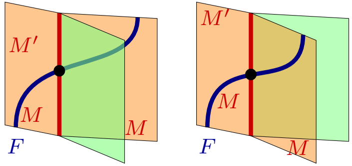
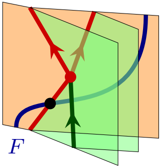


such that



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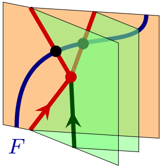

etc.

Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

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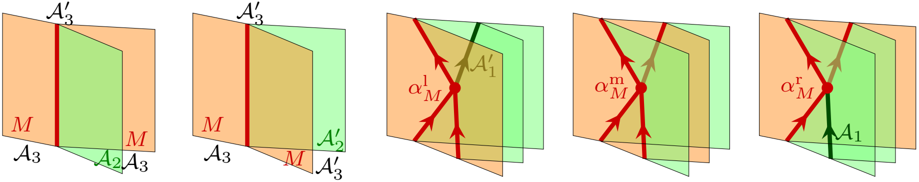

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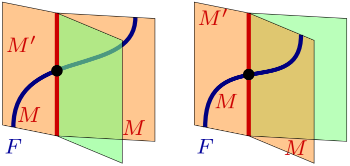

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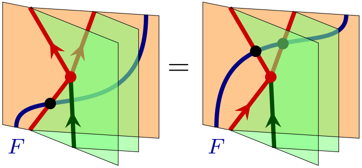
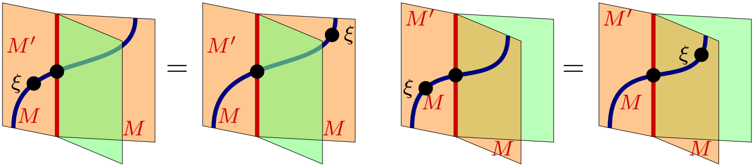


etc.

A **modification** $\xi\colon \mathcal{F} \longrightarrow \mathcal{F}'$ is $\xi\colon F \longrightarrow F'$ such that



Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

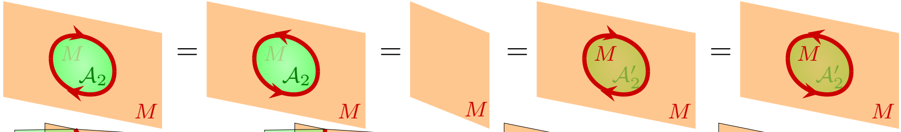
3-dimensional orbifold completion

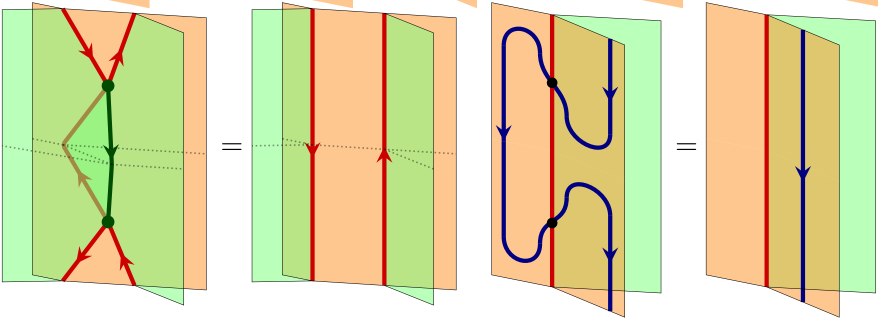
(all Hom 2-categories of $\mathcal{T}$ must admit finite sifted 2-colimits that commute with composition)

The **orbifold completion** $\mathcal{T}_{\mathrm{orb}}$ of a 3-category with duals $\mathcal{T}$ has

- objects: orbifold data
- 1-morphisms: bimodules
- 2-morphisms: maps of bimodules
- 3-morphisms: modifications
- compositions: relative products (computed via (2-)idempotents)

such that (among other axioms)



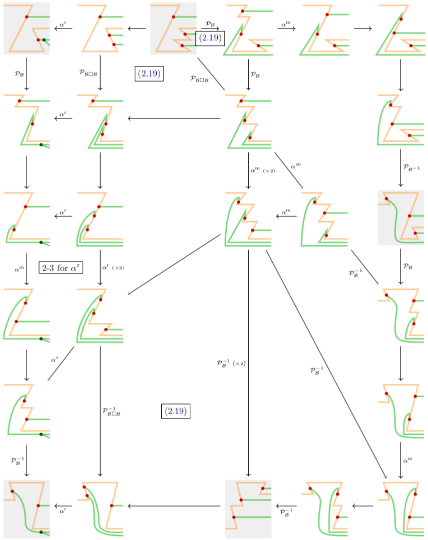


Mulevičius/Runkel 2020, Carqueville/Mulevičius/Runkel/Schaumann/Scherl 2021, Carqueville/Müller 2023

3-dimensional orbifold completion

Theorem. $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-morphisms.

Proof:



etc.

Carqueville/Müller 2023

3-dimensional orbifold completion

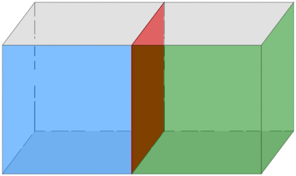
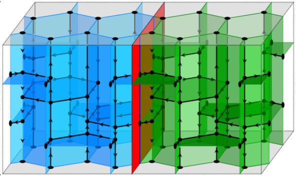
Theorem. $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-morphisms.

For $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{3,2}(\mathbb{D}) \longrightarrow \mathcal{C}$, get $\mathbb{D}^{\mathrm{orb}}$ from $(\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}}$.

Definition & Theorem. The **orbifold defect TQFT**

$$\mathcal{Z}_{\mathrm{orb}}\colon \mathrm{Bord}^{\mathrm{def}}_{3,2}(\mathbb{D}^{\mathrm{orb}}) \longrightarrow \mathcal{C}$$

is given by substratification:

$\mathcal{Z}_{\mathrm{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Carqueville/Müller 2023

(pictures courtesy of Vincentas Mulevičius)

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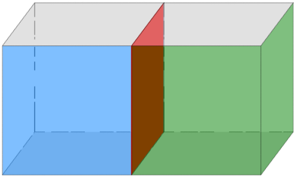
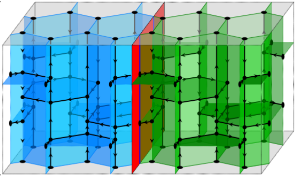
$\mathcal{Z}_{\text{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Example. *Defect state sum models are Euler completed orbifolds of the trivial defect TQFT:*

$$\mathcal{Z}^{\text{ss}}_3 \;\cong\; \big(\mathcal{Z}^{\text{triv}}_3\big)^{\odot}_{\text{orb}}$$

Carqueville/Müller 2023, Kitaev/Kong 2011, Meusburger 2022

3-dimensional orbifold completion

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Carqueville/Müller 2023, Meusburger 2022

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Theorem. Let $\mathcal{M}$ be a modular fusion category. The 3-category in which

- objects are commutative Δ-separable Frobenius algebras in $\mathcal{M}$,
- 1-morphisms from B to A are Δ-separable symmetric Frobenius algebras F over (A, B),
- 2-morphisms from F to G are G-F-bimodules M over (A, B), and
- 3-morphisms are bimodule maps

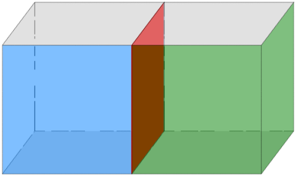
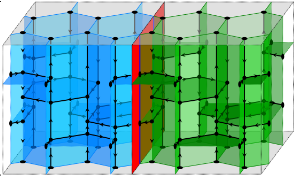
is a subcategory of $(\mathrm{B}\,\Delta\mathrm{ssFrob}(\mathcal{M}))_{\mathrm{orb}}$.

$\Longrightarrow$ recover **defect Reshetikhin–Turaev theory** à la Koppen–Mulevičius–Runkel–Schweigert

Carqueville/Müller 2023, Meusburger 2022

Summary and outlook

Theorem. $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-morphisms.

$\mathcal{Z}_{\mathrm{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Carqueville/Müller 2023

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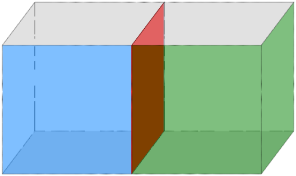
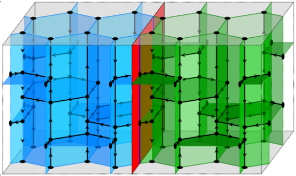
$\mathcal{Z}_{\mathrm{orb}}\Big($  $\Big) = \mathcal{Z}\Big($  $\Big)$

Get **n-dimensional trivial defect TQFT** from delooping $\mathrm{B}\,\mathcal{D}_{\mathcal{Z}^{\mathrm{ss}}_{n-1}}$.

n-dimensional defect state sum model is Euler completed orbifold of the trivial defect TQFT:

$$\mathcal{Z}^{\mathrm{ss}}_n = \left(\mathcal{Z}^{\mathrm{triv}}_n\right)^{\odot}_{\mathrm{orb}}$$

Carqueville/Müller 2023

Summary and outlook

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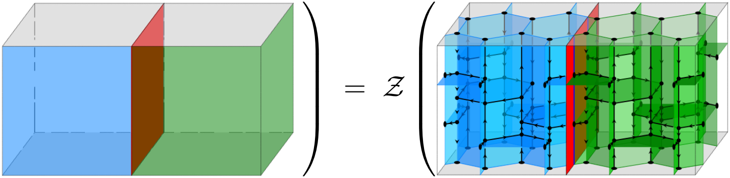

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Recover Douglas–Reutter invariants for $n = 4$.

Carqueville/Müller 2023, Carqueville/Mulevičius/Müller 20xx