Varying the non-semisimple Crane-Yetter theory over the character stack

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Goal:

- Construct a relative TQFT $Z : \mathbb{1} \Longrightarrow T$, where
  - T = once-categorified 4-d G-gauge theory
  - Z de-equivariantizes to non-semisimple Crane-Yetter.
  - can understand Z as a symmetry, that can be gauged.

$$\boxed{\bullet\ \underline{\text{Main Thm}}:\ Z \text{ is invertible relative to } T}$$

The Target

Def (sketch): The 5-category SymTens has:

objects: symmetric tensor categories $\mathcal{A}, \mathcal{B}, \dots$

1-morphisms: braided tensor categories as bimodules $\mathcal{A} \circlearrowright \mathcal{X} \circlearrowleft \mathcal{B}$

$$\longleftrightarrow \Big(\underset{\text{br. tens. cat.}}{\mathcal{X}} , \quad \mathcal{A} \boxtimes \mathcal{B}^{op} \xrightarrow[\text{sym. tens.}]{} Z_2(\mathcal{X}) \Big)$$

composition: ${}_{\mathcal{A}}\mathcal{X}_{\mathcal{B}}, \ {}_{\mathcal{B}}\mathcal{Y}_{\mathcal{C}}, \quad \mathcal{Y} \circ \mathcal{X} = \mathcal{X} \underset{\mathcal{B}}{\boxtimes} \mathcal{Y}$

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2-morphisms: tensor categories as bimodules

3-morphisms: bimodule categories

4-morphisms: functors of such

5-morphisms: natural transformations of such

Def: The 4-category BrTens is $\mathrm{BrTens} = \Omega \, \mathrm{SymTens} \simeq \mathrm{End}_{\mathrm{SymTens}}(\mathrm{Vect})$

Representations of quantum groups

Fix G reductive, q a good root of 1, $q^\ell = 1$

$$1 \longrightarrow (u_q) \longrightarrow U_q \xrightarrow{Fr} U(\mathfrak{g}) \longrightarrow 1$$

small quantum — big / Lusztig quantum group — classical

generators:

$K_i^{\pm}, E_i, F_i$

$E_i^\ell = F_i^\ell = 0$

generators: $K_i^{\pm} \longmapsto 1$

$$E_i^{(r)} \longmapsto \begin{cases} E_i^{(r/\ell)} & \ell \mid r \\ 0 & \text{else} \end{cases}$$

$$F_i^{(r)} \longmapsto \begin{cases} F_i^{(r/\ell)} & \ell \mid r \\ 0 & \text{else} \end{cases}$$

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$$\mathrm{Vect} \longleftarrow \mathrm{Rep}\, u_q \longleftarrow \mathrm{Rep}_q G \xleftarrow{Fr^*} \mathrm{Rep}\, G \longleftarrow \mathrm{Vect}$$

TQFTs from quantum groups

- $\mathrm{Rep}\, u_q$ is braided, finite, non-semisimple.

  In fact: $\mathrm{Rep}\, u_q \in \mathrm{BrTens}^{\times}$ [Brochier - Jordan - Safronov - Snyder]

  $\Rightarrow$ defines invertible 4d TQFT $\mathcal{Z} : \mathrm{Bord}_4^{fr} \to \mathrm{BrTens}$.

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- $\mathrm{Rep}\, u_q^{s.s.} \in \mathrm{BrTens}^\times \rightsquigarrow \mathcal{Z}^{ss}$: framed, extended Crane–Yetter

- We call $\mathcal{Z}$ the non-semisimple Crane–Yetter theory

- $\mathrm{Rep}_q G$ is braided, non-semisimple, non-finite, non-invertible.

$$\mathrm{Fr}^* : \mathrm{Rep}\, G \hookrightarrow Z_2(\mathrm{Rep}_q G) \qquad \left(\begin{array}{l}\text{compatible} \\ \text{module structure}\end{array}\right)$$

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$$\mathrm{Fr}^* : \mathrm{Rep}\, G \hookrightarrow Z_2(\mathrm{Rep}_q G)$$ (compatible module structure)

- Using fibre functor $\mathrm{Rep}\, G \to \mathrm{Vect}$, de-equivariantize:

$$\mathrm{Rep}_q G \boxtimes_{\mathrm{Rep}\, G} \mathrm{Vect} \simeq \mathrm{Rep}\, u_q$$ [Arkhipov - Gaitsgory, Negron]

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- Bimodule structure

$$\operatorname{Rep} G \boxtimes \operatorname{Rep} G^{op} \xrightarrow{\otimes} \operatorname{Rep} G \hookrightarrow Z_2(\operatorname{Rep}_q G)$$

$\rightsquigarrow$ Defines a 1-morphism $\operatorname{Rep} G \xrightarrow{\operatorname{Rep}_q G} \operatorname{Rep} G$ is in SymTens.

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- $\mathrm{Rep}\,G \in \mathrm{SymTens}$ is 4-dualizable

  $\Rightarrow$ defines a once-categorified 4d TQFT $T : \mathrm{Bord}_4^{fr} \to \mathrm{SymTens}$

- For M closed, $\dim M \leq 3$, $T(M) = \mathrm{QCoh}(\mathrm{Ch}_G(M))$,

$$\mathrm{Ch}_G(M) = \left[\mathrm{Hom}(\pi_1(M), G)/G\right]$$ is the character stack.

[Ben-Zvi - Francis - Nadler]

- Call T the categorified 4d G-gauge theory

$\underline{\text{Thm}}$ (K): The 1-morphism $\mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}_q G} \mathrm{Rep}\,G$ is invertible in SymTens.

- Then $\mathrm{Vect} \xrightarrow{\mathrm{Rep}\,G} \mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}_q G} \mathrm{Rep}\,G$ defines a relative theory $Z : \mathbb{1} \Rightarrow T$.

- M closed 3-manifold,

$$Z(M) = \mathcal{F} \in \mathrm{QCoh}(\mathrm{Ch}_G(M))$$

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$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathrm{Rep}\,G"] \arrow[d, squiggly] & \mathrm{Rep}\,G \arrow[r, "\mathrm{Rep}_q G"] & \mathrm{Rep}\,G \arrow[r, "\mathrm{Vect}"] & \mathrm{Vect} \simeq \mathrm{Vect} \arrow[rr, "\mathrm{Rep}\,u_q"] & & \mathrm{Vect} \arrow[d, squiggly] \\
\mathrm{Vect} \arrow[r, "\mathcal{O}"] & \mathrm{QCoh} \arrow[r, "- \otimes \mathcal{F}"] & \mathrm{QCoh} \arrow[r, "(-)_{triv}"] & \mathrm{Vect} & & \mathfrak{z}(M)
\end{tikzcd}$$

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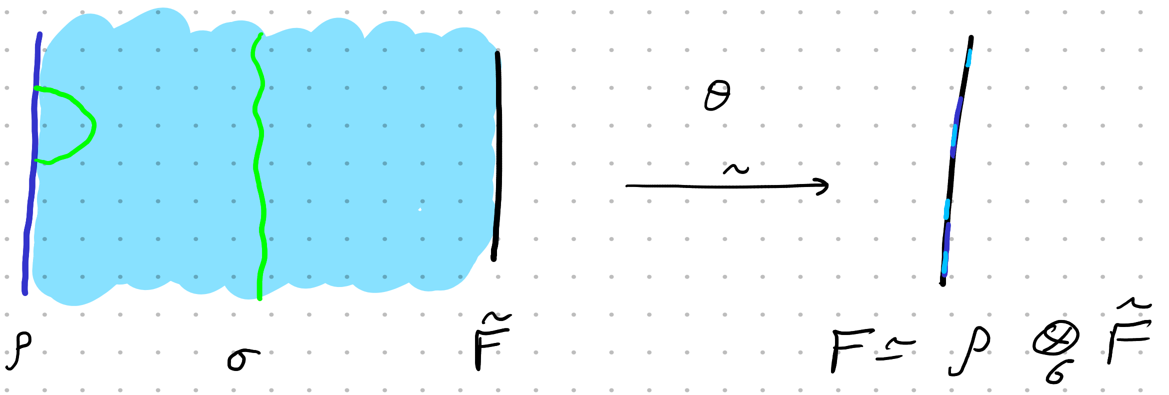
$\underline{\underline{\text{Cor}}}$: $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Ch}_G(M))$ is a line bundle.

Symmetry perspective

- Perspective of [Freed–Moore–Teleman] on symmetry:

  n-d QFT F has (σ,ρ)-symmetry if



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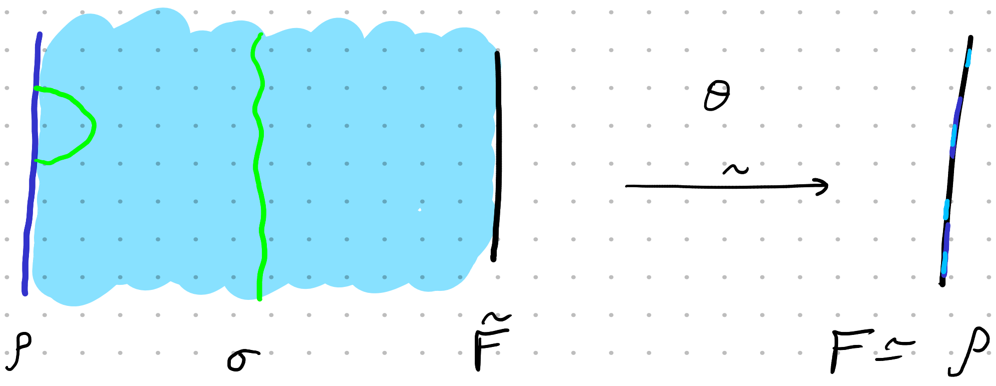


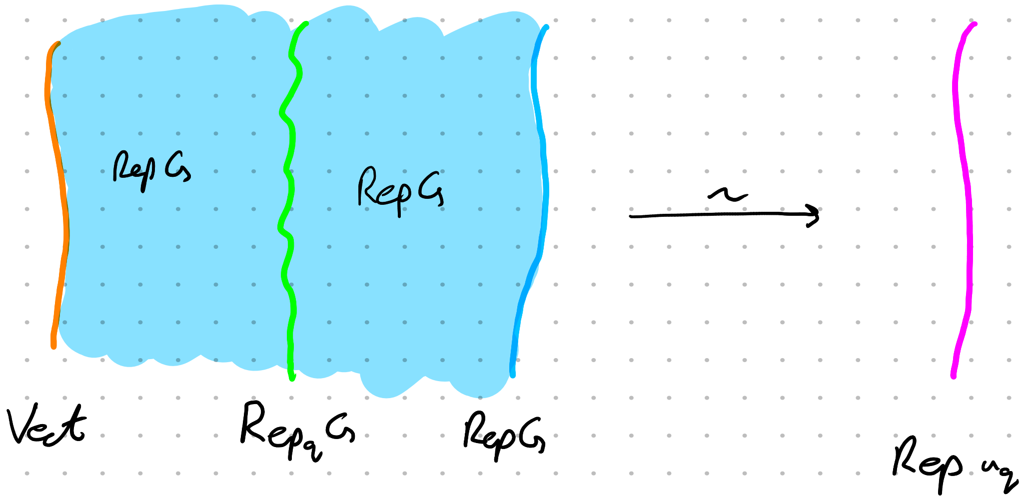
$$F \simeq \rho \otimes_\sigma \tilde{F}$$

where (σ, ρ) topological, and can support defects (e.g. boundary wall).

- <u>Quotienting/gauging</u>: if σ has a Neumann boundary condition ε, (i.e. σ-module structure on $\mathbb{1}$), the <u>gauged symmetry</u> is:

$$\varepsilon \otimes_\sigma \tilde{F}.$$

- Our setup:



$\mathrm{Rep}_q G$ a symmetry defect of 4d gauge theory, invertible <u>after</u> gauging.

<u>Thm</u> $\Rightarrow$ The defect itself is invertible.

Checking invertibility by gauging

- Let $\mathcal{C}$ be a braided tensor category, $\mathcal{A} = Z_2(\mathcal{C})$.

  Then $\mathcal{A} \boxtimes \mathcal{A}^{op} \xrightarrow{\otimes} \mathcal{A} \hookrightarrow \mathcal{C}$

  defines $\mathcal{C} : \mathcal{A} \to \mathcal{A}$ in SymTens.

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> **Thm** (K) If - $\mathcal{A}$ is semisimple, cp-rigid, has fibre funct $\mathcal{A} \to \mathrm{Vect}$
> - $\mathcal{C}$ is cp-rigid
> - $\mathcal{B} = \mathcal{C} \boxtimes_{\mathcal{A}} \mathrm{Vect}$ is finite and invertible in BrTens
>
> Then $\mathcal{C} : \mathcal{A} \to \mathcal{A}$ is invertible in Sym Tens

Our case: $\mathrm{Rep}\, G \simeq Z_2(\mathrm{Rep}_q G)$ [Negron]