String bordism invariants in dimension 3 from U(1)-valued TQFTs

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Introduction

Recalling that $\pi_n(\mathrm{String}(k)) \cong 0$ if $n \leqslant 6$ it is easy to see that

$$\mathrm{Bord}_n^{\mathrm{String}} \cong \mathrm{Bord}_n^{\mathrm{fr}} \qquad \text{for } n \leqslant 6$$

and composing with the Pontryagin–Thom isomorphism

$$\mathrm{Bord}_n^{\mathrm{fr}} \cong \pi_n \mathbb{S}$$

we get

$$\mathrm{Bord}_3^{\mathrm{String}} \cong \mathrm{Bord}_3^{\mathrm{fr}} \cong \pi_3 \mathbb{S} \cong \mathbb{Z}/24\mathbb{Z}$$

It would be nice to obtain $\varphi : \mathrm{Bord}_3^{\mathrm{String}} \xrightarrow{\cong} \mathbb{Z}/24\mathbb{Z}$ as some characteristic number given by integrating some *canonical* differential 3-form on a String 3-manifold M:

$$\varphi[M] = \int_M \omega_M.$$

This is of course impossible since the integral takes real values while φ takes values in $\mathbb{Z}/24\mathbb{Z}$ and there is no injective morphism $\mathbb{Z}/24\mathbb{Z} \to \mathbb{R}$.

The correct version of has been found by Bunke and Naumann, independently, by Redden and has recently be considered by Gaiotto, Johnson-Freyd and Witten. They consider the additional data of a triple (η, W, ∇), where:

- η is a geometric String structure on M in the sense of Waldorf;
- W is a spin 4-manifold with $\partial W = M$, which exists, since $\mathrm{Bord}_3^{\mathrm{Spin}} = 0$;
- ∇ is a spin connection on W such that the restriction $\nabla\big|_M$ coincides with the spin connection datum of the geometric String structure η.

Out of the data (M, η, W, ∇) one can compute

$$\psi(M, \eta, W, \nabla) := \frac{1}{2} \int_W \mathbf{p}_1^{CW}(\nabla) - \int_M \omega_\eta,$$

where $\mathbf{p}_1^{CW}(\nabla)$ is the Chern-Weil 4-form for the first Pontryagin class, evaluated on the connection ∇, and ω_η is the canonical 3-form associated with the geometric String structure η.

One proves that $\psi(M, \eta, W, \nabla)$ is an integer. Moreover,

$$\psi(M, \eta, W, \nabla) \mod 24$$

only depends on the string cobordism class of M.

The aim of this talk is to show how the above integral formula for ψ, as well as its main properties, naturally emerge in the context of topological field theories with values in the symmetric monoidal categories associated with morphisms of abelian groups.

Symmetric Monoidal Categories from Morphisms of Abelian Groups and TQFTs

By $\mathrm{Bord}^{\xi}_{d,d-1}(X)$ we will denote the symmetric monoidal category of $(d, d-1)$-bordism with tangential structure ξ and background fields X. That is, the objects of $\mathrm{Bord}^{\xi}_{d,d-1}(X)$ are $(d-1)$-dimensional closed manifolds M equipped with a ξ structure endowed with a map $f\colon M \to X$ to a space, or more generally smooth stack, of background fields X.

A TQFT (with tangential structure ξ and background fields X) is a monoidal functor $Z : \mathrm{Bord}^{\xi}_{d,d-1}(X) \to \mathcal{C}$.

Definition

Let $\varphi_A \colon A_{\mathrm{mor}} \to A_{\mathrm{ob}}$ be a morphism of abelian groups. By $\varphi_A^{\otimes}$ we will denote the symmetric monoidal category with

$$\begin{aligned} \mathsf{Ob}(\varphi_A^{\otimes}) &= A_{\mathrm{ob}}; \\ \mathsf{Hom}_{\varphi_A^{\otimes}}(a,b) &= \{x \in A_{\mathrm{mor}} \ : \ a + \varphi_A(x) = b\}. \end{aligned}$$

The composition of morphism is given by the sum in A_{mor}. The tensor product of objects and morphisms is given by the sum in A_{ob} and in A_{mor}, respectively. The unit object is the zero in A_{ob}. Associators, unitors and braidings are the trivial ones, i.e., they are given by the zero in A_{mor}.

When $\varphi : \mathbf{0} \to A$ is the initial map $\varphi^{\otimes}$ will be denoted by $A^{\otimes}$.

Examples

Take the stack X of background fields to be the smooth stack Ω^{d-1} of $(d-1)$-forms, and let $\mathrm{id}_{\mathbb{R}}^{\otimes}$ be the symmetric monoidal category associated with the abelian group morphism $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. Then

$$
\begin{aligned}
Z \colon \mathrm{Bord}^{\mathrm{or}}_{d,d-1}(\Omega^{d-1}) &\to \mathrm{id}_{\mathbb{R}}^{\otimes} \\
(M_{d-1}, \omega_{d-1}) &\mapsto \int_{M_{d-1}} \omega_{d-1} \\
(W_d, \omega_{d-1}) &\mapsto \int_{W_d} d\omega_{d-1}
\end{aligned}
$$

is a TQFT. With the monoidality given by the additivity of the integral and the functoriality by Stokes' theorem.

Another example is obtained by taking $X = \mathbf{B}\mathrm{U}(1)_\nabla$ and target the category associated to

$$\exp(2\pi i -) \colon \mathbb{R} \to \mathrm{U}(1).$$

There's a natural TQFT produced by these data and given by

$$\begin{aligned} Z \colon \mathrm{Bord}^{\mathrm{or}}_{2,1}(\mathbf{B}\mathrm{U}(1)_\nabla) &\to \exp(2\pi i -)^{\otimes} \\ (M_1, P, \nabla) &\mapsto \mathrm{hol}_{M_1}(\nabla) \\ (W_2, P, \nabla) &\mapsto \frac{1}{2\pi i} \int_{W_2} F_\nabla . \end{aligned}$$

where $\mathrm{hol}_{M_1}(\nabla)$ is the holonomy of ∇ and F_∇ is its curvature. Z is a TQFT thanks to the fundamental integral identity relating holonomy along the boundary and curvature in the interior:

$$\mathrm{hol}_{\partial W_2}(\nabla) = \exp\left(\int_{W_2} F_\nabla \right).$$

One defines $\mathbf{B}\mathrm{String}_{\mathrm{geom}}$ as the bottom homotopy pullback in the diagram:

$$
\begin{tikzcd}
\mathbf{B}\mathrm{String}_{\nabla} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\mathbf{B}\mathrm{String}_{\mathrm{geom}} \arrow[r, "\omega_3"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^3 \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\nabla} \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1"] & \mathbf{B}^3\mathrm{U}(1)_{\nabla}.
\end{tikzcd}
$$

The reason we prefer $\mathbf{B}\mathrm{String}_{\mathrm{geom}}$ to $\mathbf{B}\mathrm{String}_{\nabla}$ is the following theorem.

Theorem

Let M be a smooth manifold, and let $P\colon M \to \mathbf{B}\mathrm{Spin}$ be a principal spin bundle on M. Then P can be enhanced to a geometric String structure on M if and only if $\frac{1}{2}p_1(P) = 0$. That is, every String bundle admits a geometric String structure.

It is not clear if every String bundle admits a String connection, that is, if, a map to $\mathbf{B}\mathrm{String}_{\mathrm{geom}}$ can be lifted to $\mathbf{B}\mathrm{String}_{\nabla}$.

Composing with the curvature morphism $\frac{1}{2\pi i}F \colon \mathbf{B}^3\mathrm{U}(1)_\nabla \to \Omega^4_{cl}$, we obtain the commutative diagram

$$
\begin{tikzcd}
\mathbf{B}\mathrm{String}_{\mathrm{geom}} \arrow[r, "\omega_3"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^3 \arrow[d] \arrow[dr, bend left, "d"] & \\
\mathbf{B}\mathrm{Spin}_\nabla \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1"] \arrow[rr, bend right, "\frac{1}{2}\mathbf{p}_1^{CW}"'] & \mathbf{B}^3\mathrm{U}(1)_\nabla \arrow[r, "\frac{1}{2\pi i}F"] & \Omega^4_{cl}
\end{tikzcd}
$$

showing that, if $\omega_{3,M}$ is the canonical 3-form on a smooth manifold M equipped with a geometric String structure and ∇ is the underlying spin connection, the one has

$$
d\omega_{3,M} = \frac{1}{2}\mathbf{p}_1^{CW}(\nabla).
$$

Since $\mathbf{B}\mathrm{String}_{\mathrm{geom}}$ comes equipped with a morphism of smooth stacks $\mathbf{B}\mathrm{String}_{\mathrm{geom}} \xrightarrow{\omega_3} \Omega^3$ we have an associated symmetric monoidal functor $\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\Omega^3)$. Composing this with one of the $\mathrm{id}^{\otimes}_{\mathbb{R}}$-valued TQFTs above, we obtain an $\mathrm{id}^{\otimes}_{\mathbb{R}}$-valued TQFT

$$Z_{\mathrm{String}} \colon \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \to \mathrm{id}^{\otimes}_{\mathbb{R}}.$$

It maps a closed oriented 3-manifold M_3 equipped with a geometric String structure η to $\int_{M_3} \omega_{3;M}$, where $\omega_{3;M}$ is the 3-form on M_3 associated with the geometric String structure, and maps an oriented 4-manifold W_4 equipped with a geometric String structure to

$$\int_{W_4} d\omega_{3;M} = \frac{1}{2}\int_{W_4} \mathbf{p}_1^{CW}(\nabla),$$

Since $\mathbf{B}\mathrm{Spin}_\nabla$ comes equipped with a morphism $\mathbf{B}\mathrm{Spin}_\nabla \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} \mathbf{B}^3\mathrm{U}(1)_\nabla$ we have an associated monioidal functor $\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_\nabla) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}^3\mathrm{U}(1)_\nabla)$. Composing this with (a higher version of) the $\exp(2\pi i -)^{\otimes}$-valued TQFT above, we obtain a $\exp(2\pi i)^{\otimes}$-valued TQFT

$$
\begin{aligned}
Z_{\mathrm{Spin}} \colon \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_\nabla) &\to \exp(2\pi i -)^{\otimes} \\
(M_3, P, \nabla) &\mapsto \mathrm{hol}_{M_3}\left(\frac{1}{2}\hat{\mathbf{p}}_1(\nabla)\right) \\
(W_4, P, \nabla) &\mapsto \frac{1}{2\pi i}\int_{W_4} F\left(\frac{1}{2}\hat{\mathbf{p}}_1(\nabla)\right) = \int_{W_4} \frac{1}{2}\mathbf{p}_1^{CW}(\nabla).
\end{aligned}
$$

Interlude: Morphisms of Morphisms of Abelian Groups and Homotopy Fibres

If $\varphi_H\colon H_{\mathrm{mor}} \to H_{\mathrm{ob}}$ and $\varphi_G\colon G_{\mathrm{mor}} \to G_{\mathrm{ob}}$ are morphisms of abelian groups, then a morphism from φ_H to φ_G is a commutative diagram of the form

$$\begin{tikzcd} H_{\mathrm{mor}} \arrow[r, "\varphi_H"] \arrow[d, "f_{\mathrm{mor}}"'] & H_{\mathrm{ob}} \arrow[d, "f_{\mathrm{ob}}"] \\ G_{\mathrm{mor}} \arrow[r, "\varphi_G"] & G_{\mathrm{ob}} \end{tikzcd}$$

The pair $(f_{\mathrm{ob}}, f_{\mathrm{mor}})$ defines a symmetric monoidal functor $f\colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$.

We can give a simple description of the homotopy fibre of the functor $f\colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$.

Lemma

Let

$$\begin{array}{ccc} H_{\mathrm{mor}} & \xrightarrow{\varphi_H} & H_{\mathrm{ob}} \\ {\scriptstyle f_{\mathrm{mor}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{ob}}} \\ G_{\mathrm{mor}} & \xrightarrow{\varphi_G} & G_{\mathrm{ob}} \end{array}$$

be a commutative diagram of abelian groups, and let $f\colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$ *be the associated monoidal functor. Then we have*

$$\mathrm{Ob}(\mathrm{hofib}(f)) = G_{\mathrm{mor}} \times_{G_{\mathrm{ob}}} H_{\mathrm{ob}}$$

$$\mathrm{Mor}_{\mathrm{hofib}(f)}\left((g,h),(g',h')\right) = \left\{ x \in H_{\mathrm{mor}} \ \ s.t. \ \ \begin{cases} f_{\mathrm{mor}}(x) = g' - g \\ \varphi_H(x) = h' - h \end{cases} \right\}.$$

Lemma

A commutative diagram of abelian groups of the form

$$\begin{tikzcd} H_{\mathrm{mor}} \arrow[r, "\varphi_H"] \arrow[d, "f_{\mathrm{mor}}"'] & H_{\mathrm{ob}} \arrow[d, "f_{\mathrm{ob}}"] \arrow[dl, "\lambda"'] \\ G_{\mathrm{mor}} \arrow[r, "\varphi_G"] & G_{\mathrm{ob}} \end{tikzcd}$$

induces a symmetric monoidal functor $\Xi\colon \mathrm{hofib}(f) \to \ker(\varphi_G)^{\otimes}$ *acting on the objects as* $(g, h) \mapsto g - \lambda(h)$. *Moreover,* Ξ *is an equivalence iff* φ_H *is an isomorphism.*

The Bunke–Naumann–Redden Morphism

The projection $\mathbf{B}\mathrm{String}_{\mathrm{geom}} \to \mathbf{B}\mathrm{Spin}_{\nabla}$ induces a symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla})$$

and the commutative diagram of abelian groups

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"] \arrow[d, "\mathrm{id}_{\mathbb{R}}"'] & \mathbb{R} \arrow[d, "\exp(2\pi i-)"] \\ \mathbb{R} \arrow[r, "\exp(2\pi i-)"] & \mathrm{U}(1) \end{tikzcd}$$

induces a symmetric monoidal functor

$$(\mathrm{id}_{\mathbb{R}}, \exp(2\pi i-))\colon \mathrm{id}_{\mathbb{R}}^{\otimes} \to \exp(2\pi i-)^{\otimes}.$$

Lemma

The diagram of symmetric monoidal functors

$$\begin{tikzcd}
\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \arrow[r, "Z_{\mathrm{String}}"] \arrow[d] & \mathrm{id}_{\mathbb{R}}^{\otimes} \arrow[d, "{(\mathrm{id}_{\mathbb{R}},\exp(2\pi i-))}"] \\
\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla}) \arrow[r, "Z_{\mathrm{Spin}}"] & \exp(2\pi i-)^{\otimes}
\end{tikzcd}$$

commutes, with identity 2-cell.

By the naturality of the hofiber construction this implies we have a distinguished symmetric monoidal functor

$$\begin{tikzcd}
\mathrm{hofib}_{\mathrm{lax}}(\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla})) \arrow[d] \\
\mathrm{hofib}((\mathrm{id}_{\mathbb{R}}, \exp(2\pi i-))).
\end{tikzcd}$$

Since the commutative diagram defining the morphism $(\mathrm{id}_{\mathbb{R}}, \exp(2\pi i -))$ factors as

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"] \arrow[d, "\mathrm{id}_{\mathbb{R}}"'] & \mathbb{R} \arrow[d, "\exp(2\pi i-)"] \arrow[dl, "\mathrm{id}_{\mathbb{R}}"'] \\ \mathbb{R} \arrow[r, "\exp(2\pi i-)"'] & \mathrm{U}(1) \end{tikzcd} ,$$

we have a symmetric monoidal equivalence

$$\Xi \colon \mathrm{hofib}((\mathrm{id}_{\mathbb{R}}, \exp(2\pi i -))) \to \ker(\exp(2\pi i -))^{\otimes} = \mathbb{Z}^{\otimes},$$

Putting everything together we obtain a symmetric monoidal functor

$$Z^{\mathrm{Spin}}_{\mathrm{String}} \colon \mathrm{hofib}_{\mathrm{lax}}(\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\mathrm{geom}}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla})) \to \mathbb{Z}^{\otimes}$$

Theorem

The functor $Z^{\mathrm{Spin}}_{\mathrm{String}}$ *is the Bunke–Naumann–Redden map* ψ*.*

One can replace the first fractional Pontryagin class $\frac{1}{2}p_1$ with the first Chern class c_1. In this case one notices that the classifying space BSU of the special unitary group is the homotopy fiber of $c_1 \colon BU \to K(\mathbb{Z}, 2)$ so that SU and U enjoy the same kind of relationship as String and Spin. Then one can repeat the construction above to obtain an integral formula realizing the isomorphism

$$\mathrm{Bord}_1^{\mathrm{SU}} \cong \mathbb{Z}/2.$$

In this case the relevant vanishing bordism group is $\mathrm{Bord}_1^{\mathrm{U}} = 0$, and the relevant theorem is Gauss-Bonnet formula: for a closed oriented 2-manifold W_2 the integer $\int_{W_2} c_1(W_2)$ is even.

Another example is obtained by going one step higher in the Whitehead tower of the orthogonal group and consider the Fivebrane group instead of the String group. One has that the classifying space BFivebrane is the homotopy fiber of $\frac{1}{6}p_2\colon \mathrm{BString} \to K(\mathbb{Z}, 8)$, and one has a vanishing 7-dimensional bordism group given by $\mathrm{Bord}_7^{\mathrm{String}} = 0$. This argument is not complete: in order to make it fully work one would need to know that any principal String bundle admits a String connection and as we noticed above this is presently not clear.