Super duper vector spaces I

The higher categorical algebraic closure of $\mathbb{R}$.

Higher Structures in Functorial Field Theory,
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based on joint work in progress w. Theo Johnson-Freyd

I Motivation

A universal target for invertible TQFTs

Brown–Comenetz:

There is a spectrum/cohomology theory $I\mathbb{Q}/\mathbb{Z}$ uniquely determined by: $\pi_0 \hom_{\text{Spectra}}(X, I\mathbb{Q}/\mathbb{Z}) \cong \hom_{Ab}(\pi_0 X, \mathbb{Q}/\mathbb{Z})$

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Homotopy groups: $\pi_{-n} I\mathbb{Q}/\mathbb{Z} \cong \hom_{Ab}(\pi_n \mathbb{S}, \mathbb{Q}/\mathbb{Z}) \rightsquigarrow$ $\pi_{>0} = 0$, $\pi_0 = \mathbb{Q}/\mathbb{Z}$

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$\tau_{\geq 0} \Sigma^d I\mathbb{Q}/\mathbb{Z}$

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$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^X, \tau_{\geq 0}\Sigma^d I\mathbb{Q}/\mathbb{Z}) = \hom_{Ab}(\pi_d |\mathrm{Bord}_d^X|, \mathbb{Q}/\mathbb{Z})$$

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$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{fr}_d, \tau_{\geq 0}\Sigma^d I\mathbb{Q}/\mathbb{Z}) = \hom_{Ab}(\Omega^{fr}_d, \mathbb{Q}/\mathbb{Z})$$

fully extended framed $\Sigma^d I\mathbb{Q}/\mathbb{Z}$-valued d-dim TQFTs $\Longleftrightarrow$ stably framed $\mathbb{Q}/\mathbb{Z}$-valued d-dim bordism invariant

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Freed–Hopkins:
(The Picard symmetric monoidal d-groupoid of) $\tau_{\geq 0}\Sigma^d I\mathbb{Q}/\mathbb{Z}$ is a universal target for fully extended invertible (torsion) TQFT.

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{fr}, \tau_{\geq 0}\Sigma^d I\mathbb{Q}/\mathbb{Z}) = \hom_{Ab}(\Omega_d^{fr}, \mathbb{Q}/\mathbb{Z})$$

fully extended framed $\Sigma^d I\mathbb{Q}/\mathbb{Z}$-valued d-dim TQFTs $\iff$ stably framed $\mathbb{Q}/\mathbb{Z}$-valued d-dim bordism invariant

In low dimensions

finite-dim. super vector spaces

$d=0: \quad \tau_{\geq 0} I\mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$

$d=1: \quad \tau_{\geq 0} \Sigma I\mathbb{Q}/\mathbb{Z} \simeq s\mathrm{Line}_{\mathbb{C}}^{\simeq} = (s\mathrm{Vec}_{\mathbb{C}})^{\times}$

only $\otimes$-invertible objects & inv. morphisms

$d=2: \quad \tau_{\geq 0} \Sigma^2 I\mathbb{Q}/\mathbb{Z} \simeq (s\mathrm{Alg}_{\mathbb{C}})^{\times}$

Morita category of finite-dim. semisimple super-algebras

n	$\pi_n I\mathbb{Q}/\mathbb{Z}$
0	$\mathbb{Q}/\mathbb{Z}$
-1	$\mathbb{Z}/2$
-2	$\mathbb{Z}/2$
-3	$\mathbb{Z}/24$

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<u>Question</u> [Hopkins]: Is there a natural symmetric monoidal d-category $\mathcal{W}^d$ with $(\mathcal{W}^d)^{\times}_{\text{torsion}} = \tau_{\geq 0} \Sigma^d I\mathbb{Q}/\mathbb{Z}$?

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<u>Theorem</u> [with TJF]: For $d \geq 2$, there is a $\mathbb{C}$-linear symmetric monoidal d-category $\mathcal{W}^d$ with:

(1) $(\mathcal{W}^d)^{\times}_{\text{tor}} \cong \tau_{\geq 0} \Sigma^d I\mathbb{Q}/\mathbb{Z}$

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<u>Theorem</u> [with TJF]: For $d \geq 2$, there is a unique $\mathbb{C}$-linear symmetric monoidal d-category $\mathcal{W}^d$ with:

(1) $(\mathcal{W}^d)^{\times}_{\mathrm{tor}} \simeq \tau_{\geq 0} \Sigma^d I\mathbb{Q}/\mathbb{Z}$

(2) $\Omega^{d-1}\mathcal{W}^d := \mathrm{End}^{d-1}_{\mathcal{W}^d}(I) \simeq s\mathrm{Vec}_{\mathbb{C}}$

(aka inductive limit)

(3) $\mathcal{W}^d$ is a filtered colimit of finite semisimple symmetric n-cats.

Moreover, for every finite semisimple $\mathbb{R}$-linear symmetric-monoidal d-cat. $\mathcal{C}$:

$$\mathrm{Fun}^{\otimes}_{\mathbb{R}}(\mathcal{C}, \mathcal{W}^d) = \hom_{\mathbb{R}\text{-}\mathrm{Alg}}(\Omega^d \mathcal{C}, \mathbb{C})$$

A glimpse of $\mathcal{W}$

Intuition: $\mathcal{W}$ is a universal target for semisimple TFTs.

obj $\sim$ semisimple absolute n-dim. framed TFTs

mor $\sim$ framed domain walls

$\vdots$

as a framed 2D TQFT: Arf-Kervaire invariant

<u>Exm:</u> $\mathcal{W}^0 = \mathbb{C}$, $\mathcal{W}^1 = \mathrm{sVec}_{\mathbb{C}}$, $\mathcal{W}^2 = \mathrm{sAlg}_{\mathbb{C}}$

simples: I, f — simples: I, $\mathrm{Cliff}(1)$

no $\underbrace{\neq 0 \text{ morphism}}$ between them — $\exists$ $\underbrace{\neq 0 \text{ morphism}}$ between them

"domain wall"

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"domain wall"

$\mathcal{W}^3 \sim$ simples/iso = braided equiv. classes of non-degenerate braided superfusion categories + extra data

[pseudounitary case: lift of central charge from $\mathbb{Q}/\tfrac{1}{2}\mathbb{Z}$ to $\mathbb{Q}/12\mathbb{Z}$]

$\exists$ a $\neq 0$ morphism if the same element in a certain $\mathbb{Z}/24$-extension of the Witt group of slightly non-deg. braided cats.

[$\mathcal{W}^3$ similar to (and inspired by) a construction of Freed, Scheimbauer, Teleman]

Why $\mathrm{sVec}_{\mathbb{C}}$?

A field k is algebraically closed

$\Updownarrow$ Hilberts Nullstellensatz

$\forall \neq 0$, fin. generated k-algebra $k \to A$:

$$\begin{tikzcd} & A \arrow[dr, dashed, "\exists \text{ algebra map}"] & \\ k \arrow[ur] \arrow[rr, equal] & & k \end{tikzcd}$$

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But:

[The "polynomial eq."
$f^2 \cong I$, $X_{f,f} = -1_f 1_f$

has no solution in $\mathrm{Vec}_{\mathbb{C}}$!]

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$\leftarrow \exists$ no such symmetric functor!

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Slogan 1: $\mathrm{Vec}_{\mathbb{C}}$ is not algebraically closed!

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Slogan 1: $Vec_{\mathbb{C}}$ is not algebraically closed!

Theorem [Deligne]: For every $\neq 0$ not too large symmetric category $\mathcal{C}$ over $sVec_{\mathbb{C}}$:

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Slogan 2: $\mathrm{sVec}_{\mathbb{C}}$ is "algebraically closed"!
It is the "algebraic closure" of $\mathrm{Vec}_{\mathbb{C}}$.

In higher dimensions?

- $\mathbb{C}$ is the 0-categorical algebraic closure of $\mathbb{R}$.
- $\mathrm{sVec}_{\mathbb{C}}$ is the 1-categorical "algebraic closure" of $\mathbb{R}$.

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- $\mathbb{C}$ is the 0-categorical separable closure of $\mathbb{R}$.
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In this talk:

$\mathcal{W}^n$ is the n-categorical separable closure of $\mathbb{R}$.

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A Galois extension w. Galois group:

$\mathbb{Z}/2$

$\mathbb{Z}/2 \times B\mathbb{Z}/2$

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$\mathcal{W}^n$ is the n-categorical separable closure of $\mathbb{R}$.

In the next talk:

This is a Galois extension.
Its Galois group (ie. the n-categorical absolute Galois group of $\mathbb{R}$) looks a lot like PL, the stable piecewise-linear group.

II Towers

Karoubian towers

$\oplus$ at all levels ⟶ additive & lax-idempotent ⟵ [Gaiotto-Johnson-Freyd]

Karoubian n-category := complete n-category

n-categories enriched at the top in abelian groups.

$$\begin{tikzcd} n\mathrm{Cat}[\mathrm{Ab}] \arrow[r, bend left, "\mathrm{Kar}(-)"] \arrow[r, phantom, "\perp"] & \mathrm{KarCat}_n \arrow[l, bend left] \end{tikzcd}$$

[See also: Carqueville-Runkel-Schaumann: Orbifold completion

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Def: A (Karoubian) tower is a sequence $(c_n \in \mathcal{C}^n)_{n\geq 0}$ of pointed (Karoubian) n-categories $\mathcal{C}^n$ with pointed equivalences $\Omega_{c_n}\mathcal{C}^n \simeq \mathcal{C}^{n-1}$ $\forall n \geq 1$.

In words: A sequence of categories which deloop one another.

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Think: A thing with non-invertible cells in degree $[-\infty, 0]$:

$$\left.\left.\begin{matrix} 0\text{-cells} \\ (-1)\text{-cells} \\ (-2)\text{-cells} \\ \vdots \end{matrix} \right\} e^0 \right\} e^1 \Bigg\} e^2$$

In other words: A categorical (coconnective) loop spectrum.

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$$\mathrm{KarTowers} \longrightarrow \mathrm{Spectra} \qquad e^\bullet \longmapsto (e^\bullet)^\times := \text{retain only invertible cells}$$

Towers generalize commutative rings

<u>Observe</u>: If $\mathcal{C}^{\bullet}$ is a tower, then $\mathcal{C}^{n}$ inherits a symmetric monoidal structure from the ∞-deloopings.

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<u>Definition:</u> If R is a commutative ring, then the Karoubian n-cats $\Sigma^n R := \mathrm{Kar}(B^n R) \simeq \mathrm{Mod}^{dual}(\mathrm{Mod}^{dual}(\cdots(R)))$

trivially deloop R n times

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Note: $(\Sigma^\bullet R)^\times$ can have homotopy in negative degrees $\rightsquigarrow$ $\begin{cases} \pi_0 = R^\times \\ \pi_{-1} = \mathrm{Pic}(R) \\ \pi_{-2} = \mathrm{Br}(R) \\ \vdots \end{cases}$

Nullstellensatzianism

(From now on: "ring" will mean "comm. ring")

A field k is algebraically closed $\iff$ $\forall \neq 0$, finitely generated ring maps

$$\begin{tikzcd} & S \arrow[d, dashed] \\ k \arrow[ur] \arrow[r, equal] & k \end{tikzcd}$$

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A field K is separably closed $\iff$ $\forall \neq 0$, finite separable ring maps $\rightsquigarrow$

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Def: A map $F^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ of Karoubian towers is **fully finite** if $\forall n$ the n-functor $F^n : \mathcal{C}^n \to \mathcal{D}^n$ is fully right adjunctible.

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Def: A ring homomorphism $R \to S$ is **finite separable**, if:

(1) S is finitely generated projective as R-module

(2) S is finitely generated projective as $S \otimes_R S$-module

Def: A map $F^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ of Karoubian towers is **fully finite** if $\forall n$ the n-functor $F^n : \mathcal{C}^n \to \mathcal{D}^n$ is fully right adjunctible.

Prop in progress [w. TJF]:

(1) A ring map $R \to S$ is finite separable $\iff$ $\Sigma^\bullet R \to \Sigma^\bullet S$ is fully finite.

Nullstellensatzianism

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<u>Prop in progress [w. TJF]</u>:

(1) A ring map $R \to S$ is finite separable $\iff$ $\Sigma^\bullet R \to \Sigma^\bullet S$ is fully finite.

(2) If K is a field of $\mathrm{char}(K) = 0$ and $\Sigma^\bullet K \to \mathcal{C}^\bullet$ a map of towers. Then:
$\Sigma^\bullet K \to \mathcal{C}^\bullet$ is fully finite $\iff$ $\forall n$, $\mathcal{C}^n$ is a finite semisimple K-lin. n-cat.

[$n=2$ case follows from Douglas-Schommer-Pries-Snyder, $n=3$ from Décoppet]

Nullstellensatzianism

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Nullstellensatzianism

<u>Def:</u> A Karoubian tower $\mathcal{R}^\bullet$ is separably closed $\iff$ $\forall \neq 0$, fully finite ring maps

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<u>Exm:</u> $\Sigma^\bullet \mathbb{C}$ is not separably closed.

[cf. recent work of Burklund–Schlank–Yuan, Barthel–Carmeli–Schlank–Yanovski in chromatic homotopy theory.]

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(2) $R^\bullet \to S^\bullet$ is a filtered colimit of fully finite maps.

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Nullstellensatzianism

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(2) $\mathcal{R}^\bullet \to S^\bullet$ is a filtered colimit of fully finite maps.

Thm in progress [w. TJF]:

$\mathcal{W}^\bullet$ is the unique separable closure of $\Sigma^\bullet \mathbb{R}$.

This explains why $\mathcal{W}^\bullet$ is a filtered colimit of finite semisimple towers:

(Recall: Fully finite $\Sigma^\bullet \mathbb{R} \to \mathcal{C}^\bullet \Leftrightarrow$ finite semisimple $\mathbb{R}$-tower $\mathcal{C}$)

[cf. recent work of Burklund–Schlank–Yuan, Barthel–Carmeli–Schlank–Yanovski in chromatic homotopy theory.]

III Constructing W

Roots of unity

<u>Recall:</u> K a separably closed field of char. 0.

Then: $\mu(K) := \begin{matrix}\text{roots of unity}\\ \text{in } K\end{matrix} = (K^\times)_{tor} \cong \mathbb{Q}/\mathbb{Z} \in Ab.$

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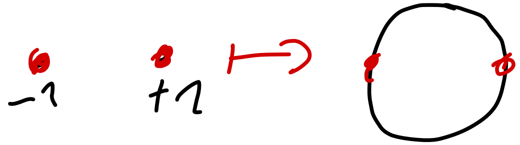
<u>Idea:</u> Build $\mathcal{W}$ by adjoining missing roots of unity.

<u>Difficulty:</u> Don't adjoin too many!
- $\to$ products of new roots might be roots.
- $\to$ sums of new roots might be roots

Building $\mathbb{C}$

want these roots of unity

Look at $\quad \mu(\mathbb{R}) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$



$$0 \longrightarrow \mu(\mathbb{R}) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{2\cdot} \underbrace{\mathbb{Q}/\mathbb{Z}}_{\text{missing roots}} \longrightarrow 0$$

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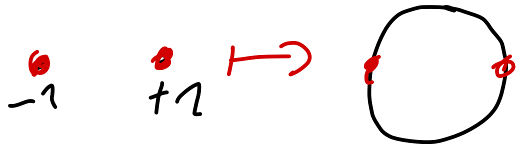

$$\omega \in \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/2)$$

$$\Updownarrow$$

$$\mathbb{Q}/\mathbb{Z} \xrightarrow{2\cdot} \underbrace{\mathbb{Q}/\mathbb{Z}}_{\text{missing roots}} \xrightarrow{\omega} \Sigma H\mathbb{Z}/2$$

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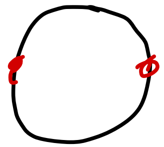
Look at $\mu(\mathbb{R}) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$

$-1 \quad +1 \mapsto$ 

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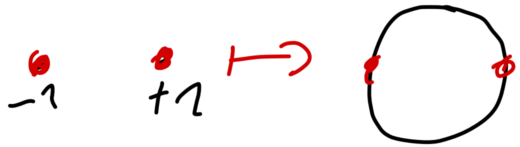

$$\begin{tikzcd} & A \arrow[d, hook] & \\ \mathbb{Q}/\mathbb{Z} \arrow[r, two heads, "2\cdot"] & \underbrace{\mathbb{Q}/\mathbb{Z}}_{\text{missing roots}} \arrow[r, "\omega"] & \mathbb{Z}/2[1] \end{tikzcd}$$

$$\mathbb{R}^{\omega|A}[A]$$

Building $\mathbb{C}$

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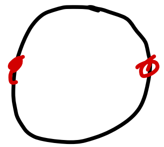
$-1 \quad +1 \mapsto$ 

$$
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\end{tikzcd}
$$

$$\mathbb{R}^{w/A}[\mathbb{Z}/2] \cong \mathbb{C}$$

Building $\mathcal{W}^n$

<u>Notation:</u> For a symmetric n-cat. $\mathcal{C}$, $\mathrm{Tor}(\mathcal{C}) := \{\text{torsion objects in } \mathcal{C}\}/\mathrm{iso}$

$$\mu(\Sigma^{\bullet}\mathbb{C}) \longrightarrow I\,\mathbb{Q}/\mathbb{Z}$$

iso on π_0: $\mu(\mathbb{C}) \longrightarrow \mathbb{Q}/\mathbb{Z}$

mono on π_{-1}: $\underbrace{\mathrm{Tor}(\Sigma\mathbb{C})}_{:= \pi_{-1}\mu(\Sigma^{\bullet}\mathbb{C}) = \{\text{torsion objects of } \Sigma\mathbb{C} = \mathrm{Vec}_{\mathbb{C}}\}/\mathrm{iso}} = 0 \to \mathbb{Z}/2)$

Building W^n

<u>Notation:</u> For a symmetric n-cat. $\mathcal{C}$, $\mathrm{Tor}(\mathcal{C}) := \{\text{torsion objects in } \mathcal{C}\}/\mathrm{iso}$

<u>Induction:</u>

$$\mu(\Sigma^\bullet W^{n-1}) \longrightarrow I\,\mathbb{Q}/\mathbb{Z}$$

iso on $\pi_{\geq -(n-1)}$

mono on π_{-n}

Building $\mathcal{W}^n$

<u>Notation:</u>
- For a symmetric n-cat. $\mathcal{C}$, $\mathrm{Tor}(\mathcal{C}) := \{\text{torsion objects in } \mathcal{C}\}/\mathrm{iso}$
- $\pi^n := \pi_{-n}$

<u>Induction:</u>

$$\mu(\Sigma^{\bullet}\mathcal{W}^{n-1}) \longrightarrow I\,\mathbb{Q}/\mathbb{Z} \qquad \begin{array}{l}\text{iso on } \pi^{\leq(n-1)} \\ \text{mono on } \pi^n\end{array}$$

$$\mu(\Sigma^{\bullet}\mathcal{W}^{n-1}) \to I\,\mathbb{Q}/\mathbb{Z} \to \underbrace{C}_{\text{cofiber w. } \pi^{\leq(n-1)} = 0} \to \mu(\Sigma^{\bullet}\mathcal{W}^{n-1})[1]$$

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$$\begin{tikzcd}
 & & F^n := \pi^n C \arrow[d] \arrow[dr, "\omega_n"] & \\
\mu(\Sigma^\bullet \mathcal{W}^{n-1}) \arrow[r] & I\,\mathbb{Q}/\mathbb{Z} \arrow[r] & \underbrace{C}_{\text{cofiber w. } \pi^{\leq (n-1)} = 0} \arrow[r] & \mu(\Sigma^\bullet \mathcal{W}^{n-1})[1]
\end{tikzcd}$$

Building W^n

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- For a symmetric n-cat. $\mathcal{C}$, $\mathrm{Tor}(\mathcal{C}) := \{\text{torsion objects in } \mathcal{C}\}/\text{iso}$
- $\pi^n := \pi_{-n}$

<u>Induction:</u> $\mu(\Sigma^\bullet W^{n-1}) \longrightarrow I\,\mathbb{Q}/\mathbb{Z}$ iso on $\pi^{\leq(n-1)}$, mono on π^n

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<u>Unpacked:</u> $0 \to \mathrm{Tor}(\Sigma W^{n-1}) \to \hom(\pi_n \mathbb{S}, \mathbb{Q}/\mathbb{Z}) \to F^n \to \mathrm{Tor}(\Sigma^2 W^{n-1}) \to \hom(\pi_{n+1}\mathbb{S}, \mathbb{Q}/\mathbb{Z}) \to \text{junk}$

<u>Think:</u> F^n
- a) contain el's of $\mathrm{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times)$ that aren't already in $\mathrm{Tor}(\Sigma W^{n-1})$.
- b) kill el's of $\mathrm{Tor}(\Sigma^2 W^{n-1})$ that aren't in $\mathrm{Hom}(\pi_{n+1}\mathbb{S}, \mathbb{C}^\times)$

Building $\mathcal{W}^n$

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Induction:

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$\underbrace{C}$ cofiber w. $\pi^{\leq(n-1)} \cong 0$

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Think: F^n
- a) contain el's of $\mathrm{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times)$ that aren't already in $\mathrm{Tor}(\Sigma \mathcal{W}^{n-1})$,
- b) kill el's of $\mathrm{Tor}(\Sigma^2 \mathcal{W}^{n-1})$ that aren't in $\mathrm{Hom}(\pi_{n+1}\mathbb{S}, \mathbb{C}^\times)$

Def: $\mathcal{W}^n := (\Sigma \mathcal{W}^{n-1})^{\omega_n}[F^n]$ $\quad$ n-categorical twisted group algebra

Formally: $\mathrm{colim}\left[F^n \xrightarrow{\omega_n} (\Sigma^2 \mathcal{W}^{n-1})^\times_{\mathrm{tor}} \to \mathrm{Mod}(\Sigma \mathcal{W}^{n-1})\right] = \left(\bigoplus_{f \in F_n} \omega_n(f)\right)$

In other words: $\mathcal{W}^n$ is an F^n-graded extension of $\Sigma \mathcal{W}^{n-1}$.

Building $\mathcal{W}$

<u>To build:</u> $\mu(\Sigma^\bullet \mathcal{W}^n) \to I\mathbb{Q}/\mathbb{Z}$ is iso on $\pi^{\leq n}$, mono on π^{n+1}

Building $\mathcal{W}$

<u>To build:</u> $\mu(\Sigma^\bullet \mathcal{W}^n) \to I_{\mathbb{Q}/\mathbb{Z}}$ is iso on $\pi^{\leq n}$, mono on π^{n+1}

<u>For $n \geq 1$:</u> Consider $X := \mathrm{cofib}\left(\mu(\Sigma^\bullet \mathcal{W}^{n-1}) \to \mu(\Sigma^\bullet \mathcal{W}^n)\right)$

<u>Unpacked:</u> $0 \to \mathrm{Tor}(\Sigma \mathcal{W}^{n-1}) \to \mathrm{Tor}(\mathcal{W}^n) \to \pi^n X \to \mathrm{Tor}(\Sigma^2 \mathcal{W}^{n-1}) \to \mathrm{Tor}(\Sigma \mathcal{W}^n)$

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<u>Unparked:</u> $0 \to \mathrm{Tor}(\mathcal{E}\mathcal{W}^{n-1}) \to \mathrm{Tor}(\mathcal{W}^n) \to F^n \to \mathrm{Tor}(\mathcal{E}^2\mathcal{W}^{n-1}) \twoheadrightarrow \mathrm{Tor}(\mathcal{E}\mathcal{W}^n)$

because $\mathcal{W}^n = (\mathcal{E}\mathcal{W}^{n-1})^{\omega_n}[F^n]$ is a <u>categorial</u> group algebra

because $\mathcal{W}^{n-1}$ is separably closed.

Building $\mathcal{W}$

<u>To build:</u> $\mu(\Sigma^\bullet \mathcal{W}^n) \to I\mathbb{Q}/\mathbb{Z}$ is iso on $\pi^{\le n}$, mono on π^{n+1}

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because $\mathcal{W}^{n-1}$ is separably closed.

This implies:

$$
\begin{tikzcd}
\mu(\Sigma^\bullet \mathcal{W}^{n-1}) \arrow[r] \arrow[d] & I\mathbb{Q}/\mathbb{Z} \\
\mu(\Sigma^\bullet \mathcal{W}^n) \arrow[ur, dashed] &
\end{tikzcd}
$$

iso on $\pi^{\le n}$
mono on π^{n+1}

By induction, get $\mathcal{W}^\bullet$ with $\mu(\mathcal{W}^\bullet) \simeq I\mathbb{Q}/\mathbb{Z}$.