Reflection positivity for extended topological field theories

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based on joint work with Luuk Stehouwer (arXiv:2301.06664) & Dagger Higher Categories workshop: Bruce Bartlett, Gio Ferrer, Brett Hungar, Theo Johnson-Freyd, Cameron Krulewski, Nivedita Nivedita, Dave Penneys, David Reutter, Claudia Scheimbauer, Luuk Stehouwer, Chetan Vuppulury

Functorial field theories as a model for quantum field theory?

Two pillars of quantum field theory:

- Locality: Objects only influence their immediate surroundings
  - Compatibility with cutting and gluing
  - $\Rightarrow$ Extended functorial field theories.
- Unitarity: probabilities are positive and add up to 1
  - State spaces are Hilbert spaces
  - Orientation reversal corresponds to complex conjugation
  - $\Rightarrow$ Unitary/ Reflection positive functorial field theories [Freed Hopkins '21, Kontsevich Segal '21,...]

Open Question

How to define reflection positive **extended** *functorial field theories?*

Restriction to TFTs

Goal for today

Study reflection positivity for once extended topological field theories.

Freed Hopkins '21:

- Definition for non-extended TQFTs in terms of hermitian structures and $\mathbb{Z}_2$-equivariant functors for a large class of tangential structures.
- Definition and classification for fully extended invertible field theories.

Kreck, Stolz, Teichner:

- Definition for non-extended TQFTs in terms of $\dagger$-functors.
- Classification for non-extended invertible field theories.

1. Reflection and spin-statistics topological field theories
2. Higher dagger categories
3. Reflection positivity for extended topological field theories

Symmetries and fermionic groups

Symmetries in physics:

- Fermionic systems always have a preferred symmetry $(-1)^F$.
- Distinction between *time preserving* and *time reversing* symmetries.

Definition

A *fermionic group* G is a $\mathbb{Z}_2$-graded Lie group $G = G_0 \sqcup G_1$ together with an even central element $c \in G$ of square 1.

Example

Let V be a super vector space. The group

$$\mathrm{Aut}^f(V) := \mathrm{Aut}(V)_0 \sqcup \mathrm{Iso}(\overline{V}, V)_1$$

is a fermionic group with $c = (-1)^F$.
(A *representation of a fermionic group* G on a super vector space is a fermionic group homomorphism $G \longrightarrow \mathrm{Aut}^f(V)$.)

Spacetime structure groups

Definition

For G a fermionic group we define the *spacetime structure group* $H_d := (Pin_d^+ \rtimes G/\{(-1,1) \simeq (1,c)\})_{\mathrm{ev}}$ with group structure $(p \times g) \cdot (p' \times g') = c^{\theta(g)\theta(p')} \cdot (pp' \times gg')$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & BH_d & \longrightarrow & BH_{d+1} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & BO_d & \longrightarrow & BO_{d+1} & \longrightarrow & \cdots \end{array}$$

Example

- $G = * \Rightarrow H_d = \mathrm{SO}_d$
- $G = \mathbb{Z}_2^c \Rightarrow H_d = \mathrm{Spin}_d$, $G = \mathbb{Z}_2^T \Rightarrow H_d = \mathrm{O}_d$,
- $G = \mathbb{Z}_n \Rightarrow H_d = \mathrm{SO}_d \times \mathbb{Z}_n$
- $G = \mathrm{Pin}_1^+ \Rightarrow H_d = \mathrm{Pin}_d^-$, $G = \mathrm{Pin}_1^- \Rightarrow H_d = \mathrm{Pin}_d^+$

$\mathbb{Z}_2 \times B\mathbb{Z}_2$-action on $\mathrm{Bord}_d^{H_d}$

$$BH_d \longrightarrow B\widehat{H}_d \longrightarrow B\mathbb{Z}_2$$

induces a map $B\mathbb{Z}_2 \to \mathrm{Top}_{/BO_d}$. In addition $c \in H_d$ induces a homotopy

$$BH_d \;\underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}}\; BH_d \qquad (c \colon \mathrm{id} \Rightarrow \mathrm{id})$$

They combine into a $\mathbb{Z}_2 \times B\mathbb{Z}_2$-action on BH_d as a space over BO_d.
Using functionality of $\mathrm{Bord}_d^{(-)} \colon \mathrm{Top}_{/BO_d} \to \mathrm{Cat}$.

Spin as an example

We consider $G = \mathbb{Z}_2^c$ as an example

$$1 \to \mathrm{Spin}_d \to \mathrm{Pin}_d^+ \to \mathbb{Z}_2 \to 1$$

- $(P \to M) \longmapsto \overline{P} := \widehat{P} \setminus P$ with $\widehat{P} = P \times_{\mathrm{Spin}_d} \mathrm{Pin}_d^+$.
- $B\mathbb{Z}_2$ acts via the natural transformation with components

$$\begin{array}{ccc} P & \xrightarrow{\quad \cdot c \quad} & P \\ & \searrow \qquad \swarrow & \\ & M & \end{array}$$

Reflection structures and spin statistics for topological field theories

- $\mathbb{Z}_2 \times B\mathbb{Z}_2$ also acts on sVect by complex conjugation and $(-1)^F : \mathrm{id} \Longrightarrow \mathrm{id}$ with component

$$(-1)^F_V : V_0 \oplus V_1 \xrightarrow{\mathrm{id}_{V_0} \oplus -\mathrm{id}_{V_1}} V_0 \oplus V_1$$

Definition

Let G be a fermionic group with associated space time structure group H_d. A *reflection and spin-statistics topological field theory with internal G-symmetry* is a $\mathbb{Z}_2 \times B\mathbb{Z}_2$-equivariant symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_d^{H_d} \longrightarrow \mathrm{sVect} \quad .$$

Reflection structures and spin statistics for topological field theories

- The equivariance data consists of natural isomorphisms $\mathcal{Z}(\overline{\Sigma}) \cong \overline{\mathcal{Z}(\Sigma)}$ and the condition that $\mathcal{Z}(c_\Sigma) = (-1)^F_{\mathcal{Z}(\Sigma)}$.

Proposition (LM L Stehouwer '23)

1-D reflection and spin-statistics topological field theory with internal G-symmetry are equivalent to hermitian representations of $\pi_0(G)$.

Proof.

$\bullet [\mathrm{Bord}_1^{H_1}, \mathrm{sVect}]^{\mathbb{Z}_2 \times B\mathbb{Z}_2} \cong ((\iota_0 \mathrm{sVect})^{H_1})^{\mathbb{Z}_2 \times B\mathbb{Z}_2} \cong (\iota_0 \mathrm{sVect})^{H_1 \rtimes (\mathbb{Z}_2 \times B\mathbb{Z}_2)}$

$\bullet H_1 \rtimes (\mathbb{Z}_2 \times B\mathbb{Z}_2) \cong G_b \times O_1$

$\bullet \implies [\mathrm{Bord}_1^{H_1}, \mathrm{sVect}]^{\mathbb{Z}_2 \times B\mathbb{Z}_2} = ((\iota_0 \mathrm{sVect})^{\mathbb{Z}_2})^{G_b} \cong (\iota_0 \mathrm{sHerm})^{G_b}$ □

Reflection positivity for topological field theories

Lemma

There are isomorphisms $h_\Sigma \colon \overline{\Sigma} \to \Sigma^\vee$ *satisfying*

$$(\overline{\Sigma} \xrightarrow{\sim} \overline{\Sigma^{\vee\vee}} \xrightarrow{\overline{h_\Sigma^\vee}} \overline{\overline{\Sigma^\vee}} \xrightarrow{\sim} \Sigma^\vee) = \overline{\Sigma} \xrightarrow{h_\Sigma} \Sigma$$

for all elements $\Sigma \in \mathrm{Bord}_d^{H_d}$.

$\Rightarrow \overline{\mathcal{Z}(\Sigma)} \cong \mathcal{Z}(\overline{\Sigma}) \xrightarrow{\mathcal{Z}(h_\Sigma)} \mathcal{Z}(\Sigma^\vee) \cong \mathcal{Z}(\Sigma)^\vee$ defines a hermitian pairing on $\mathcal{Z}(\Sigma)$.

Definition (Freed Hopkins '21)

A reflection topological field theory $\mathcal{Z}$ is *positive* if all the hermitian structures $\mathcal{Z}(h_\Sigma)$ are positive, i.e. define a super Hilbert space.

Choosing h_Σ is evil!

Reformulation in terms of †-functors

- We can use the choices of h_Σ to make $\mathrm{Bord}_d^{H_d}$ into a dagger category

$$\dagger\colon \mathrm{Bord}_d^{H_d} \longrightarrow (\mathrm{Bord}_d^{H_d})^{\mathrm{op}}$$
$$M\colon \Sigma \to \Sigma' \longmapsto M\colon \Sigma' \xrightarrow{\text{“}h_{\Sigma'}\text{”}} \overline{\Sigma'}^{\vee} \xrightarrow{\overline{M}^{\vee}} \overline{\Sigma}^{\vee} \xrightarrow{\text{“}h_{\Sigma}\text{”}} \Sigma \quad .$$

- Similarly, sHilb is a dagger category.
- Reflection positivity $\iff$ $\mathcal{Z}$ induces a symmetric monoidal †-functor [L. Stehouwer J. Steinebrunner '23, Stehouwer]

Theorem (Stehouwer)

Every reflection positive field theory automatically satisfies spin-statistics.

Extended theories

- The functoriality of the extended bordism category $\mathrm{Bord}_{d,0}^{H_d}$ in H_d implies that it also carries a $\mathbb{Z}_2 \times B\mathbb{Z}_2$-action.
- Various higher categories build from sVect also carry a $\mathbb{Z}_2 \times B\mathbb{Z}_2$-action:
  - The Morita category $\mathrm{Alg}_1(\mathrm{sVect})$ and higher iterations $\mathrm{Alg}_1(\dots \mathrm{Alg}_1(\mathrm{Alg}_1(\mathrm{sVect}))\dots)$
  - The n-fold suspensions $\mathrm{nsVect} = \Sigma^{n-1}\mathrm{sVect}$ of Gaiotto and Johnson-Freyd.
  - Semi-simple super categories $\mathrm{sCat} \cong \mathrm{sAlg}^{f.d.}$.
  - The universal target of Johnson-Freyd and Reutter.

Definition

Let $\mathcal{C}$ be any of the targets listed above. A *fully extended reflection and spin-statistics topological field theory with internal G-symmetry* is a $\mathbb{Z}_2 \times B\mathbb{Z}_2$-equivariant symmetric monoidal functor

$$\mathcal{Z}\colon \mathrm{Bord}_{d,0}^{H_d} \longrightarrow \mathcal{C} \quad .$$

Dagger $(\infty, 1)$-categories

Definition

An *anti-involutive* $(\infty, 1)$*-category* is a fixed point for the action $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ on $\mathrm{Cat}_{(\infty,1)}$.

Definition

A *dagger structure* on an $(\infty, 1)$-category $\mathcal{C}$ is a anti-involutive structure, together with a fully faithful ∞-subgroupoid $\mathcal{C}_0 \hookrightarrow (\iota_0\mathcal{C})^{\mathbb{Z}/2}$ such that the induced map $\mathcal{C}_0 \hookrightarrow (\iota_0\mathcal{C})^{\mathbb{Z}/2} \to \mathcal{C}$ is essentially surjective.

Example

$\mathrm{Bord}_d^{H_d}$ with anti-involutive structure $(M \colon \Sigma \to \Sigma') \longmapsto (\overline{M}^\vee : \overline{\Sigma'}^\vee \to \overline{\Sigma}^\vee)$ and $\mathcal{C}_0$ consisting of the full subcategory of $(\iota_0\mathrm{Bord}_d^{H_d})^{\mathbb{Z}_2}$ on the objects (Σ, h_Σ).

Dagger (∞, n)-categories

Theorem (C. Barwick and C. Schommer-Pries '21)

The automorphism group of the ∞-groupoid of (∞, n)-categories is $\mathbb{Z}_2^n$.

Definition

A *fully anti-involutive (∞, n)-category* is a fixed point for the $\mathbb{Z}_2^n$-action on $\mathrm{Cat}_{(\infty,n)}$.

Definition

A *dagger (∞, n)-category* is a collection of (∞, n)-category

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n$$

such that each $\mathcal{C}_k$ is a (fully-)anti-involutive (∞, k)-category, and the map $\mathcal{C}_k \to \mathcal{C}_{k+1}$ is a map of (fully-)anti-involutive $(\infty, k+1)$-categories (where $\mathcal{C}_k$ is given the trivial $(k+1)$th anti-involution) which is essentially surjective on $(\leq k)$-morphisms and induces a map $\mathcal{C}_k \to (\iota_k \mathcal{C}_{k+1})^{\mathbb{Z}/2}$ which is fully-faithful on $(> k)$-morphisms.

Dagger bicategories

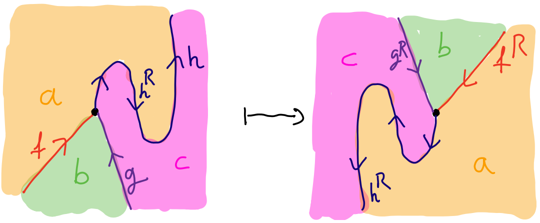
- Effectively $\mathcal{C}_0$ picks out preferred fixed point structures for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\iota_0 \mathcal{C}$ induced from the fully anti-involutive bicategory $\mathcal{C}_2$.
- $\mathcal{C}_1$ picks out preferred $\mathbb{Z}_2$-fixed point structures on 1-morphisms compatible with those on objects.
- We can build an equivalent bicategory $\mathcal{C}_2'$ with $\dagger_1 \colon \mathcal{C}_2' \to \mathcal{C}_2'^{\,1\,\mathrm{op}}$ and $\dagger_2 \colon \mathcal{C}_2' \to \mathcal{C}_2'^{\,2\,\mathrm{op}}$ such that
  - $\dagger_1$ is the identity on objects and comes with an natural isomorphism $\dagger_1^2 \Rightarrow \mathrm{id}$ whose components at every object are the identity.
  - $\dagger_2$ is the identity on objects and 1-morphisms and $\dagger_2^2 = \mathrm{id}$.

Fully dualizable n-categories

- The categories showing up in the study of topological field theories are rigid, i.e. admit adjoints for all morphisms.
- There should be graphical calculus in terms of framed stratifications of $\mathbb{R}^n$ for rigid n-categories.

Conjecture (T. Johnson-Freyd)

The automorphism group of $\mathrm{RigidCat}_{(\infty,n)}$ *is* PL_n.



PL_n-dagger categories

- A PL_n *anti-involutive* (∞, n)*-category* is a fixed point for the PL_n-action on $\mathrm{RigidCat}_{(\infty,n)}$.
- A PL_n-dagger (∞, n)-category

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n$$

consists of PL_k anti-involutive (∞, k)-categories $\mathcal{C}_k$ such that $\mathcal{C}_i \to \mathcal{C}_{i+j}$ is $\mathrm{PL}_i \times \mathrm{PL}_j$-equivariant (+similar conditions as above).

Examples from topological field theories with defects

Example

Let $\mathcal{C}$ be a fully dualizable symmetric monoidal (∞, n)-category. The stratified cobordism hypothesis describes defects in unoriented fully extended topological field theories in terms of

$$(\iota_0\mathcal{C})^{O_n} \to (\iota_1\mathcal{C})^{O_{n-1}} \to \cdots \to \mathcal{C}$$

This should be an O_n-dagger (∞, n)-category.

What can we actually prove?

There is an $\mathrm{PL}_2 = \mathrm{O}_2 = S^1 \rtimes \mathbb{Z}_2 \simeq B\mathbb{Z} \rtimes \mathbb{Z}_2$ on RigidBiCat given by:

- $\mathcal{B} \longmapsto \mathcal{B}^{2\,\mathrm{op}} \cong \mathcal{B}^{1\,\mathrm{op}}$
- $\mathrm{id}_{\mathrm{RigidBiCat}} \Longrightarrow \mathrm{id}_{\mathrm{RigidBiCat}}$ with component at $\mathcal{B}$:

$$(-)^{RR} \colon \mathcal{B} \longrightarrow \mathcal{B}, \quad b \longmapsto b$$
$$f \longmapsto f^{RR}, \varphi \longmapsto \varphi^{RR}$$

- $\cdots$

Remark

SO_2-dagger categories are the same as pivotal bicategories. This provides an explanation for their appearance in oriented topological field theories.

Symmetric monoidal bicategories

- We will now restrict to once extended topological field theories $\mathcal{Z} \colon \mathrm{Bord}_{d,d-2}^{H_d} \to \mathrm{sAlg}$.
- Symmetric monoidal higher dagger categories can be defined by requiring that all the $\mathcal{C}_i$ and functors in the definition are symmetric monoidal.

Observation (D. Reutter)

Let $\mathcal{B}$ be a fully dualizable symmetric monoidal bicategory. It has a canonical (symmetric monoidal) O_2-anti-involutive structure. All other (symmetric monoidal) O_2-anti-involutive structures differ by an O_2-action.

$(-)^\vee \colon \mathcal{B} \longrightarrow \mathcal{B}^{1\,\mathrm{op}}$, $S \colon \mathrm{id}_{\mathcal{B}} \Longrightarrow (-)^{RR}$, ...
Restricting this O_2-anti-involution to $\iota_0\mathcal{B}$ recovers the O_2-action featuring in the cobordism hypothesis.

O_2-dagger structure on $\mathrm{sAlg}^{f.d.}$

- We consider the O_2-anti-involutive structure on sCat twisted by the action featuring in the definition of reflection and spin statistics field theories via the map $O_2 \to B\mathbb{Z}_2 \times \mathbb{Z}_2$.
- The $\mathbb{Z}_2$ part of the action is $\mathcal{C} \longmapsto \overline{\mathcal{C}}^{\vee} \cong \overline{\mathcal{C}}^{\mathrm{op}}$. $(F^{\vee} \cong (F^L)^{op})$
- The SO_2-part is $S_{\mathcal{C}} \circ (-1)^F_{\mathcal{C}}$ where

$$S_{\mathcal{C}} = N^r_{\mathcal{C}} = \int^{d \in \mathcal{C}} \mathcal{C}(c, d)^* \otimes d$$

- O_2-hermitian structures on $\mathcal{C}$:
    - $h_{\mathcal{C}} \colon \overline{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C}$ together with isomorphisms $h^2_{\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}}$ and coherences.
    - 2-isomorphisms $\lambda \colon S_{\mathcal{C}} \circ (-1)^F \Longrightarrow \mathrm{id}_{\mathcal{C}}$. $\Leftrightarrow \{\mathrm{tr}_c \colon \mathrm{End}(c) \to \mathbb{C}\}_{c \in \mathcal{C}}$, i.e. a super Calabi-Yau structure.
    - Coherences...

O_2-dagger structure on $\mathrm{sAlg}^{f.d.}$

We want to pick out some preferred O_2-hermitian structures:

- Restrict $h_{\mathcal{C}}$ to be a 1-categorical dagger structure $\dagger_{\mathcal{C}}$.
- $\implies \langle f, g \rangle := \mathrm{tr}_c(f^\dagger \circ g)$ for $f, g \in \mathcal{C}(c, c')$ defines a non-degenerate pairing.
- We further restrict to those for which this pairing is positive.
- This is a super version of Baez's 2-Hilbert spaces.

Theorem

Baez super 2-Hilbert spaces s2Hilb *form an O_2-dagger category.*

O_2-dagger structure on $\mathrm{Bord}_{2,0}^{H_2}$

- We consider the O_2-anti-involutive structure on $\mathrm{Bord}_{2,0}^{H_2}$ twisted by the action featuring in the definition of reflection and spin statistics field theories via the map $O_2 \to B\mathbb{Z}_2 \times \mathbb{Z}_2$.
- The $\mathbb{Z}_2$ part of the action is $P \longmapsto \overline{P}^{\vee}$.
- The SO_2-part is $S_P \circ c_P$.
- We fix O_2-hermitian structures on P:
  - $h_P \colon \overline{P}^{\vee} \to P$ as in the non-extended case.
  - There is a path $\gamma \colon 1 \to c$ in H_2 which is mapped to a generating loop in SO_2.
  - $\implies$ ($\mathrm{Bord}_{2,0}^{H_2} = (\mathrm{Bord}_{2,0}^{*})_{H_2}$) There is a 2-isomorphism $S_P \circ c_P \to \mathrm{id}_P$.
  - Coherences...

O_2-dagger structure on $\mathrm{Bord}_{2,0}^{H_2}$

- On 1-morphisms $\Sigma\colon P \to P'$ hermitian structures consist of 2-isomorphisms $h_\Sigma\colon (h_P \circ \overline{\Sigma}^\vee \circ h_{P'}^{-1})^R \to \Sigma$

Theorem

This defines a symmetric monoidal O_2-dagger category structure on $\mathrm{Bord}_{2,0}^{H_2}$.

2-dimensional fully extended reflection positive TFTs

Definition

A *2-dimensional fully extended reflection positive topological field theory with internal symmetry* G is a symmetric monoidal O_2-dagger functor $\mathcal{Z}\colon \mathrm{Bord}_{2,0}^{H_2} \to \mathrm{s2Hilb}$.

Theorem

They are classified by unitary 2-representation of $\Pi_1(G)$.

A unitary 2-representation on a super 2-Hilbert space H consists of

- For even elements $g \in G_0$ unitary functors $\rho(g)\colon H \to H$
- For odd elements $g \in G_1$ unitary functors $\rho(g)\colon \overline{H} \to H$
- For every homotopy class of paths $\gamma\colon g \to g'$ a natural unitary isomorphism $\rho(\gamma)\colon \rho(g) \Longrightarrow \rho(g')$

Outline of the proof

• Classify anti-involutive symmetric monoidal functors $\mathrm{Bord}_{2,0}^{H_2} \to \mathrm{sCat}$ [LM L. Stehouwer '23]:

- They are equivalent to $\mathbb{Z}_2 \times B\mathbb{Z}_2$-equivariant functors.
- Use the extension $1 \to H_2 \to H_2 \rtimes (\mathbb{Z}_2 \times B\mathbb{Z}_2) \to \mathbb{Z}_2 \times B\mathbb{Z}_2 \to 1$ and the isomorphism $H_2 \rtimes (\mathbb{Z}_2 \times B\mathbb{Z}_2) \cong G_b \times O_2$.
- The O_2-fixed points $\iota_0(\mathrm{sCat})^{O_2}$ are exactly the O_2-hermitian structures.
- The G_b-fixed points provide the action.

• Being an O_2-dagger functor forces these to come from 2-super Hilbert spaces and unitary functors.

Outlook and questions

- Is there an extended spin statistics theorem ($\mathbb{Z}_2$-dagger implies O_2-dagger)?
- Comparison with the classification of Freed-Hopkins in the invertible case. (work in progress with L. Stehouwer)
- Fully extended reflection positivity:
  - What is a good target?
  - Is $\mathrm{Bord}_{d,0}^{H_d}$ a PL_n or O_n dagger category?

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Thank you for your attention!