Seminar on functorial field theory

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Introduction

In this seminar, we will discuss the definition of functorial field theory, with a focus on examples relevant to differential geometry. In the plan below, the first (shorter) part of the seminar is about topological field theories (i.e., functorial field theories whose values do not depend on geometric input data on the input), the second is about geometric field theories, which are defined on suitable geometric versions of the bordism category.

Talks

Part 1: Topological field theories

(1) Definition of topological quantum field theory. The talk should roughly follow §2 of [CR18]. Give the path integral motivation (§2.1, see also and then define topological field theories (TFTs) as functors, as in [CR18, §2.2]. The classical source is [Ati88] (people often say that Atiyah's paper was inspired by Witten and typically cite [Wit88]). Give first properties as in §2.4 & §2.5 of [CR18].

(2) Examples of TFTs Classify 1-dimensional topological field theories as in [CR18, §3.2]. Discuss some examples of higher-dimensional topological field theories: Euler theory, finite gauge theory [Fre19, Example 1.23], Dijkgraf-Witten theory [Fre19, Example 1.44 & 1.50], characteristic numbers [Fre19, Example 1.62], Kervaire theory [Fre19, Example 6.15], Hopf theory [Fre19, Example 6.17]. For field theories with super vector spaces as target, see [Fre19, Example 1.57 & 1.59].

(3) Classification of 2-dimensional TFTs. Review the classification of 2-dimensional topological field theories as commutative Frobenius algebras, which already goes back to Atiyah [Ati97]. A possible source is [CR18, §3.3]. This could be contrasted with Thm. 3.52 of the thesis [SP09] of Schommer-Pries, which is about *extended* 2-dimensional topological field theories (see also the unoriented version, Thm. 3.54). It should be interesting for later to at least include a sketch of the bicategory of algebras, bimodules and intertwiners.

Part 2: Geometric field theories

(4) Geometric cobordism categories. Discuss the following motivating examples: Quantum mechanics and parallel transport [Fre19, Example 2.46]. Give a definition of a bordism category of Riemannian spin manifolds, rougly following [LR20, §3.1] and/or [Lud23a, Definition 4.7]. Indicate how to build in smoothness, following [LS21, §3.6]. Here the notion of a general geometry should be specialized to, say Riemannian metrics, possibly with spin structure. The purpose of the latter is not to achieve a working knowledge of the definition of the smooth geometric bordism category, but to get some feeling for the difficulties that arise.

(5) Classical Chern-Simons theory. Explain classical Chern-Simons theory in the formalism of functorial field theory, see [Fre95, §2]. This is an invertible 3-dimensional field theory defined on manifolds with principal G-bundle with connection, for a fixed Lie group G. Its definition is given in Thm. 2.19.

(6) Wess-Zumino-Witten theory. Wess-Zumino-Witten theory is the so-called *boundary theory* associated to Chern-Simons-theory; it is a 2-dimensional theory on manifolds with a map to a Lie group G. The talk should follow [Fre95, Appendix A].

(7) & (8) The free boson. The purpose of this talk is to give a description of the free boson as a geometric functorial field theory, motivated by the path integral. The reference is the thesis and paper by Kandel [Kan14, Kan16]. Start by introducing the Dirichlet-to-Neumann map ([Kan16, §2.1], see also [Kan14, §5.1]) and present the heuristic discussion of §3.1 in [Kan16]. The full field theory is the composition of two functors: A functor to the category of Lagrangian correspondences (end of §2.1 in [Kan16]) and a further second quantization functor to the category of Hilbert spaces and Hilbert-Schmidt maps (§3.3).

(9) The first-quantized free fermion. The free fermion is defined on Riemannian manifolds with spin structure. The purpose of this talk is a description of a "pre-second-quanitzed" free fermion functor from the corresponding bordism category to the category of Lagrangian correspondences, as in [Lud23b] (notice the difference to the one appearing in the previous talk, where a similar category was considered corresponding to *symplectic* vector spaces, which are here replaced by Hilbert spaces). Start by sketching the definition of the category Lagrangian correspondences as in [Lud23b, §3]. Then explain the definition of the field theory as in §4 of [Lud23b].

(10) The second quantization anomaly. One would now like to postcompose the field theory constructed in the previous talk with a second-quantization functor, to obtain the free fermion. However, this functor is "anomalous", which mathematically means that it takes values in Hilbert spaces with *projective* Hilbert-Schmidt transformations. The goal of this talk is to describe this projective field theory in the 2-dimensional case, following [Ten17, §4]; see also [Ten14]. The "anomaly problem" can be overcome using a more fancy concept, namely that of a *twisted field theory*. End this talk by introducing this notion following [ST11, §5.1] and briefly sketch the relation to the problem at hand.

(11) The anomaly theory I. The anomaly theory (or *twist*) of the d-dimensional free fermion is (conjecturally) the $(d-1, d)$-dimensional part of a $(d+1)$-dimensional invertible field theory. The purpose of this talk is to describe the $(d, d+1)$-dimensional part, following Dai and Freed [DF94] (see also [DF01]). This theory assigns to a $(d+1)$-dimensional (Riemannian, spin^c) manifolds the exponentiated eta invariant of its Dirac operator, and to d-dimensional manifolds the determinant line. Start by defining the exponentiated eta invariant (denoted by τ_X) on manifolds X with boundary for suitable boundary conditions [DF94, §1] and its transformation formula under change of boundary conditions (Thm. 1.4). Conclude that, without specifying boundary conditions, τ_X is an element of the line $L_{\partial X}$ defined in (1.7). Then introduce the determinant line of spin^c-manifolds [DF94, §2] and identify these with the lines $L_{\partial X}$ (Prop. 2.15). The functoriality of this assignment follows from Thm. 2.20. If time permits, highlight why it is important that this theory takes values in *super* vector spaces and not just vector spaces.

(12) The anomaly theory II. The purpose of this talk is the description of the $(d-1, d)$-dimensional part of the (conjectural) anomaly theory of the free fermion described above. This takes values in the bicategory of algebras, bimodules and intertwiners. The reference is [LR20], with the construction given in §3.3. As the field theory takes values in a bicategory, functoriality only holds in a weak sense (isomorphism instead of equality of bimodules), involving a higher coherence condition. Functoriality is discussed after Lemma 3.20, and the coherence is Thm. 3.22.

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