Seminar "Coarse geometry and topological Phases"

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Part I: Coarse Geometry

0. (8.4.2025) Crash Course on C^*-Algebras and K-Theory. — (If needed) Introduction to the basic concepts of the theory of C^*-algebras and their K-theory, as well as the exact six-term sequence.

1. (22.4.) Coarse Spaces — Introduction of the concepts of *(proper) coarse structure*, *coarse map*, *coarse equivalence* [13, §2.1], [2, §6.1], [12, §2]. Possibly a comparison with concepts from metric geometry [13, §1.3]. As an example, maybe a discussion of the Milnor-Švarc theorem (Thm. 1.18 in [13]).

2. (29.4.) Roe Algebras — Definition of the *Roe algebra* of a metric space [12, §3], [2, §6], [1, §2]. Discussion of "functoriality" under coarse maps and independence from the choice of the X-module [2, Lemma 6.3.11 & Prop. 6.3.12], [1, Thm. 2.1]. Discussion of the theorem stating that the K-theory of the Roe algebra vanishes on *flasque* spaces [12, Def. 9.3 & Prop. 9.4], [2, Lemma 6.4.2], [3, Prop. 1], [1, Prop. 3.9].

3. (6.5.) The Coarse Mayer-Vietoris Sequence — Discussion of *big families* and *localized Roe algebras* [4, III.5 & III.6]. Subsequently, introduction of the *Mayer-Vietoris sequence* for coarse K-theory [12, §9], [3, §5], [1, §3.4]. In the literature, this is done for "coarsely excisive partitions" [12, Def. 9.1]. A more practical approach is to formulate the result for big families instead (i.e., replace subsets with the generated big family), thus avoiding additional conditions and obtaining the MV-sequence in the form given in [4, §V]. Computation of the K-theory groups $K_i(C^*(\mathbb{R}^d)) \cong K_i(C^*(\mathbb{Z}^d))$ through repeated applications of the Mayer-Vietoris sequence and the flasqueness of half-spaces [1, §3.8].

4.* (13.5.) Comparison with the Group C^*-Algebra — Definition of the *(reduced) group C^*-algebra $C^*_r(\Gamma)$* of a discrete group Γ. For $\Gamma = \mathbb{Z}^d$, discussion of the isomorphism $C^*_r(\mathbb{Z}^d) \cong C(\mathbb{T})$ and computation of the K-theory. Definition of the map $C^*_r(\mathbb{Z}^d) \otimes \mathbb{K} \to$

$C^*(\mathbb{R}^d)$ and computation of the induced map in K-theory [1, §4] (here, it suffices to restrict to the real case).

Part II: Topological Phases

5. (20.5.) Lattice Systems and Chern Insulators — Discussion of translation-invariant Hamiltonians on lattice systems and their classification over the *Brillouin zone* via Berry curvature. Connection to the mathematical theory of vector bundles and characteristic classes. Explicit description of a *Chern insulator* [4, Ex. IV.1].

6. (27.5.) Noncommutative Approach — Description of a topological insulator using C^*-algebras [4, §I]. Relation to the translation-invariant setting using material from Lecture 4. Discussion of examples, in particular the Landau Hamiltonian $H_A = (d - iA)^*(d - iA)$ in $\mathbb{R}^2$, including the statement that this yields a nontrivial class [5], see also Ex. 5.3 in [6] and the references therein.

7. (27.5.) Gap-Filling — If a Hamiltonian with a spectral gap describes a topological insulator, then the spectral gap disappears after restriction to a bounded region. Discussion of this *gap-filling* principle following [4, §VI], see also [5, Thm. 2], [6, Thm. 3.4].

8. (3.6.) Edge-Traveling — For two-dimensional topological insulators, a boundary current can be observed after introducing an edge. Description of this *edge-traveling* phenomenon following [4, §VII] (see also [6, §6], though this description is more complicated). For experimental observations, see the videos related to the article [10].

9. (10.6.) Localized Wannier Bases — Discussion and proof of the statement that the spectral subspace defined by nontrivial topological insulators does not admit an orthonormal basis of localized functions [7].

10. (17.6.) Coarse Cohomology — Introduction of *coarse cohomology* [11, §2.2], see also §4.2 in [14] (considering only the non-equivariant case). Discussion of the Mayer-Vietoris sequence for coarse cohomology and its computation in the case of $\mathbb{R}^n$. Possibly also a discussion of the character map $HX^\bullet(M) \to H_c^\bullet(M)$ [11, 2.11]. Proof that for a coarsely transverse partition $A_0, \dots, A_q$ (see Def. 2.5 in [8]), the function

$$\varphi(x_0, \dots, x_n) := \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \chi_{A_0}(x_{\sigma_0}) \cdots \chi_{A_q}(x_{\sigma_q})$$

defines a coarse cohomology class on M in degree n, where χ_{A_i} is the indicator function of A_i.

11.* (24.6.) "Measurement" via Coarse Cohomology — By choosing cohomology classes appropriately, one can "measure" the nontriviality of insulators (i.e., determine whether they are topological). This is treated in dimension 2 in [8]. The pairing of a finite propagation projection with a partition is defined in §2.3, and the quantization argument is discussed in §2.1 and §2.5. For simplicity, it is advisable to restrict to finite propagation projections here. Possibly a discussion of relation to physics (§5.4). For physical applications, see the videos related to [9].

References

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